Gröbner Bases for Families of Affine or Projective Schemes

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Abstract

Let I be an ideal of the polynomial ring $A[x] = A[x_1, \dots, x_n]$ over the commutative, noetherian ring A. Geometrically I defines a family of affine schemes over $\mathrm{Spec}(A)$: For $\mathfrak{p} \in \mathrm{Spec}(A)$, the fibre over $\mathfrak{p}$ is the closed subscheme of affine space over the residue field $k(\mathfrak{p})$, which is determined by the extension of I under the canonical map $\sigma_\mathfrak{p} : A[x] \to k(\mathfrak{p})[x]$. If I is homogeneous there is an analogous projective setting, but again the ideal defining the fibre is $\langle \sigma_\mathfrak{p}(I) \rangle$. For a chosen term order this ideal has a unique reduced Gröbner basis which is known to contain considerable geometric information about the fibre. We study the behavior of this basis for varying $\mathfrak{p}$ and prove the existence of a canonical decomposition of the base space $\mathrm{Spec}(A)$ into finitely many locally closed subsets over which the reduced Gröbner bases of the fibres can be parametrized in a suitable way.

Introduction

Let A be a commutative, noetherian ring with identity and $A[x] = A[x_1, \dots, x_n]$ the polynomial ring in the variables $x_1, \dots, x_n$ over A. We denote the residue field at $\mathfrak{p} \in \mathrm{Spec}(A)$ by $k(\mathfrak{p})$. Geometrically an ideal $I \subset A[x]$ defines a family of affine schemes over $\mathrm{Spec}(A)$: The canonical map $A \to A[x]/I$ gives rise to a morphism of affine schemes

$$\varphi : \mathrm{Spec}(A[x]/I) \to \mathrm{Spec}(A).$$

For $\mathfrak{p} \in \mathrm{Spec}(A)$ the fibre $\varphi^{-1}(\mathfrak{p})$ is the closed subscheme of $\mathbb{A}^n_{k(\mathfrak{p})} = \mathrm{Spec}(k(\mathfrak{p})[x])$ determined by $\langle \sigma_\mathfrak{p}(I) \rangle$ where $\sigma_\mathfrak{p} : A[x] \to k(\mathfrak{p})[x]$ denotes the trivial extension of the canonical map $A \to k(\mathfrak{p})$.

If I is a homogeneous ideal we analogously obtain a family of projective schemes from

$$\varphi : \mathrm{Proj}(A[x]/I) \to \mathrm{Spec}(A).$$

The fibre $\varphi^{-1}(\mathfrak{p})$ is the closed subscheme of $\mathbb{P}^n_{k(\mathfrak{p})} = \mathrm{Proj}(k(\mathfrak{p})[x])$, again determined by $\langle \sigma_\mathfrak{p}(I) \rangle$.

For a chosen term order we wish to study – simultaneously for all $\mathfrak{p} \in \mathrm{Spec}(A)$ – the unique reduced Gröbner basis of $\langle \sigma_\mathfrak{p}(I) \rangle$. It is well known that such a

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Gröbner basis facilitates "easy access" to geometric information about the fibre $\varphi^{-1}(\mathfrak{p})$. It also seems reasonable to compare two fibres by "comparing" the corresponding Gröbner bases. Of course we can compare the leading terms, however it is not quite clear what comparing the Gröbner bases should mean. We will make this notion precise by introducing parametric sets. Rather vaguely a parametric set with respect to I is a locally closed subset Y of $\operatorname{Spec}(A)$ such that over Y the reduced Gröbner bases of the fibres can be parameterized in a suitable way. The main result of this article is to establish the existence and uniqueness of a canonical decomposition of the base space $\operatorname{Spec}(A)$ into finitely many parametric sets.

Many concrete mathematical problems can be stated in the above described framework of families of affine or projective schemes and to know the Gröbner basis structure of the fibres may be the first step to their solution, if not yet the solution itself. For example, if A is a polynomial ring over some field, then we obtain the case of algebraic systems with parameters, which is important for many "real life" applications such as robotics or electrical engineering (see e.g. [6], chapter 6, and [18]). From a more theoretical point of view parametric sets are a tool to explore the geometry of families of affine or projective schemes. Related theoretical applications range from efficient Gröbner basis computation (see e.g. [2] and [19]) to cohomology (see [20]).

The naive hope that for a Gröbner basis G of I the specialized Gröbner basis $\sigma_{\mathfrak{p}}(G)$ is a Gröbner basis of the specialized ideal $\langle\sigma_{\mathfrak{p}}(I)\rangle$ is in general not fulfilled. The behavior of Gröbner bases under specialization (or extension of scalars) has actually been studied by many authors, e.g. [5], [14], [3], [9] [10], [4], [8]. In [3] the case of standard bases in the ring of formal power series is treated. Relations to flatness are explored in [4] and also in [5]. Articles focusing more on the fibres are [21], [22], [17] and [16]. These last articles were written from a more computational point of view, which led to a rather rash use of the word "canonical". So one main objective of the present article is to establish a proper theoretical foundation for the underlying ideas of these articles.

The outline of the article is the following: Section 1 *(Parametric sets)* introduces the fundamental notion of parametric sets and their basic properties. The main theorem of section 2 *(Lucky primes and pseudo division)* is a characterization of parametric sets in terms of lucky primes (see [11]). This theorem can also be understood as giving the geometric meaning of luckiness. Finally in section 3 *(Gröbner covers)* we achieve the main objective of the article by proving existence and uniqueness of a canonical finite covering of $\operatorname{Spec}(A)$ with parametric subsets.

Preliminaries and notation

A parametric subset Y of $\operatorname{Spec}(A)$ facilitates an object which parameterizes the reduced Gröbner bases of $\langle\sigma_{\mathfrak{p}}(I)\rangle$ for $\mathfrak{p} \in Y$. To assure uniqueness of this object, which will be called the reduced Gröbner basis of I over Y we have to work with reduced schemes $(Y, \mathcal{O}_Y)$. In particular we would like to assume that our base ring A is reduced. This can be done without loss of generality:

Let $\operatorname{Nil}(A)$ denote the nilradical of A and define $A' = A/\operatorname{Nil}(A)$. Then there

is a natural homeomorphism

$$\begin{aligned} \operatorname{Spec}(A) &\to \operatorname{Spec}(A') \\ \mathfrak{p} &\mapsto \mathfrak{p}' \end{aligned}$$

and $k(\mathfrak{p}) = k(\mathfrak{p}')$. Moreover if $I' \subset A'[x]$ denotes the extension of I under the canonical map $A[x] \to A'[x]$ then $\langle \sigma_{\mathfrak{p}}(I) \rangle = \langle \sigma_{\mathfrak{p}'}(I') \rangle$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

Throughout A denotes a commutative, noetherian, reduced ring with identity and I an ideal of the polynomial ring $A[x] = A[x_1, \ldots, x_n]$. For an A-module M the localization at $\mathfrak{p} \in \operatorname{Spec}(A)$ is denoted by $M_{\mathfrak{p}}$ and $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is the residue field at $\mathfrak{p}$. The map $\sigma_{\mathfrak{p}} : A[x] \to k(\mathfrak{p})[x]$ denotes the coefficientwise extension of the canonical map $A \to k(\mathfrak{p})$.

We will only consider reduced subschemes of $\operatorname{Spec}(A)$. So by a subscheme of $\operatorname{Spec}(A)$ we mean a locally closed subset Y of $\operatorname{Spec}(A)$ with the induced reduced subscheme structure $\mathcal{O}_Y$. Let $\mathfrak{a}$ be the radical ideal of A with $\overline{Y} = \mathrm{V}(\mathfrak{a})$. (As usual $\mathrm{V}(\mathfrak{a}) \subset \operatorname{Spec}(A)$ denotes the closed set of all prime ideals containing $\mathfrak{a}$.) For an open set U of Y we can explicitly describe $\mathcal{O}_Y(U)$ as the set of all functions from U into the disjoint union $\coprod (A/\mathfrak{a})_{\mathfrak{p}}$ which are locally fractions. We will continuously identify $\operatorname{Spec}(A/\mathfrak{a})$ with $\mathrm{V}(\mathfrak{a}) \subset \operatorname{Spec}(A)$.

The set of terms (i.e. powerproducts) is denoted by $\mathcal{T} = \mathcal{T}(x_1, \ldots, x_n)$. Throughout we fix a term order $<$ on $\mathcal{T}$. For a nonzero polynomial $P = \sum_{t \in \mathcal{T}} a_t t \in A[x]$ we define

- the *coefficient of* P *at* t by $\operatorname{coef}(P, t) = a_t$,
- the *support of* P by $\operatorname{supp}(P) = \{t \in \mathcal{T};\ a_t \neq 0\}$,
- the *leading term* $\operatorname{lt}(P)$ *of* P to be the maximal element of $\operatorname{supp}(P)$,
- the *leading coefficient of* P by $\operatorname{lc}(P) = \operatorname{coef}(P, \operatorname{lt}(P))$ and
- the *leading monomial of* P by $\operatorname{lm}(P) = \operatorname{lc}(P) \operatorname{lt}(P)$.

For $G \subset A[x]$ we set $\operatorname{lt}(G) = \{\operatorname{lt}(P);\ P \in G \setminus \{0\}\}$ and similarly $\operatorname{lm}(G) = \{\operatorname{lm}(P);\ P \in G \setminus \{0\}\}$. A finite subset G of I is called a Gröbner basis of I if $\langle \operatorname{lm}(G) \rangle = \langle \operatorname{lm}(I) \rangle$. For $t \in \mathcal{T}$ we define the ideal of leading coefficients at t by

$$\operatorname{lc}(I, t) = \{\operatorname{lc}(P);\ P \in I \text{ with } \operatorname{lt}(P) = t\}.$$

Note that $\operatorname{lc}(I, t)$ can conveniently be read off from a Gröbner basis G of I. In fact, $\operatorname{lc}(I, t)$ is generated by $\{\operatorname{lc}(g);\ g \in G \text{ with } \operatorname{lt}(g) \text{ divides } t\}$. For a general reference for Gröbner bases over rings see [1].

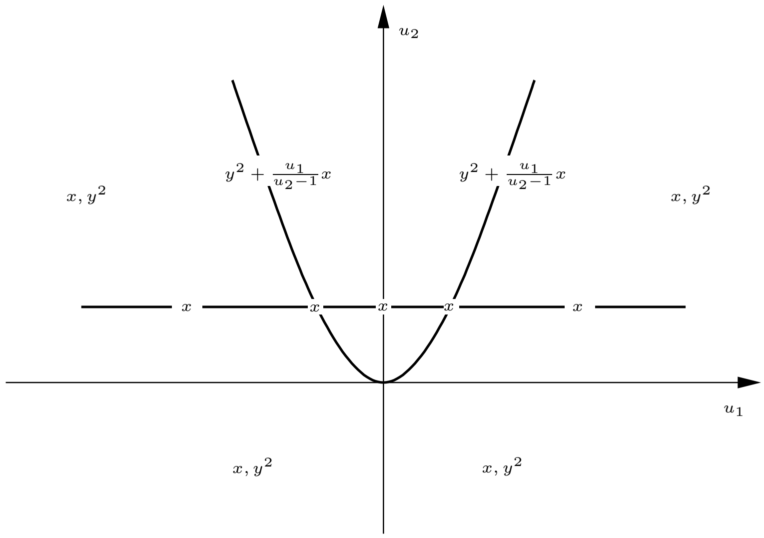
Before really getting started we look at some warm-up examples:

Example 1. Let k be a field and $A = k[u_1, u_2]$ the polynomial ring in the two parameters u_1, u_2. Consider the ideal

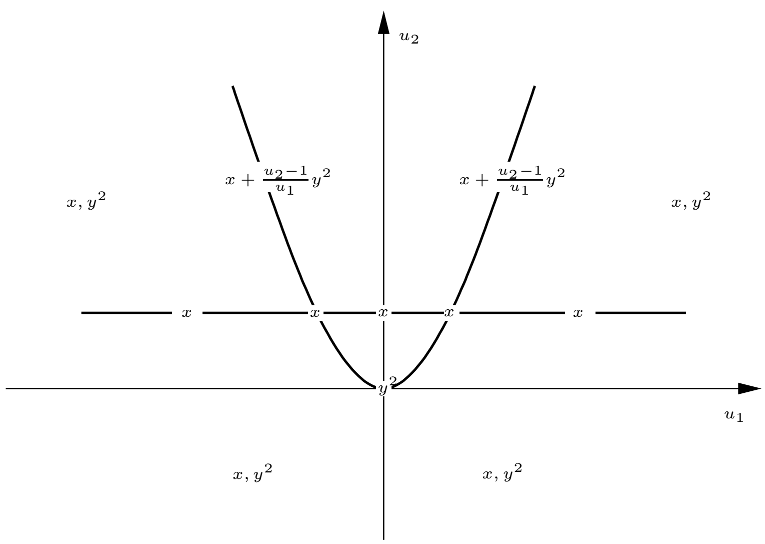
$$I = \left\langle (u_1^2 - u_2)x, (u_2 - 1)y^2 + u_1 x \right\rangle \subset A[x, y].$$

When faced with the task to describe the Gröbner basis structure of the fibres I guess most mathematicians would come up with the following pictures:

term order with $y^2 > x$:



term order with $x > y^2$:



The above pictures illustrate a decomposition of the base space $\mathbb{A}_k^2 = \mathrm{Spec}(A)$ into locally closed subsets. In short, the objective of this article is to find this decomposition in general.

Example 2. Let k be an algebraically closed field and $A = k[u_1, u_2, u_3, u_4]$ the polynomial ring in the parameters u_1, u_2, u_3, u_4. We consider the ideal

$$I = \left\langle (u_2u_3 - u_4u_1)x,\ u_1x^2 + u_2x,\ u_3x^2 + u_4x \right\rangle \subset A[x].$$

(Here x denotes just one variable.) Let $v = (v_1, v_2, v_3, v_4) \in k^4$ and

$$\mathfrak{p}_v = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4 \rangle.$$

If $v_2 v_3 - v_4 v_1$ is nonzero then the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}_v}(I) \rangle$ is x. If v_1 and v_3 are zero and one of v_2, v_4 is nonzero then the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}_v}(I) \rangle$ is also x. (In particular the set of all $v \in k^4$ such that $\mathrm{lt}(\langle \sigma_{\mathfrak{p}_v}(I) \rangle)$ is generated by x is not locally closed.) If v lies in the quasi-affine variety $Y = \mathrm{V}(\langle u_2 u_3 - u_4 u_1 \rangle) \smallsetminus \mathrm{V}(\langle u_1, u_3 \rangle)$ then the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}_v}(I) \rangle$ is given by $x^2 + f(v)x$ where f denotes the regular function on Y defined by

$$f(v) = \begin{cases} v_2/v_1 & \text{if} \quad v_1 \neq 0 \\ v_4/v_3 & \text{if} \quad v_3 \neq 0. \end{cases}$$

The above example illustrates the "local nature" of the problem and suggests to work with sheaves and not just with polynomials in I, as was common practice in [22] or [16].

Using the Buchberger algorithm it is relatively easy to see that the equivalence relation $\sim$ defined on $\mathrm{Spec}(A)$ by $\mathfrak{p} \sim \mathfrak{p}'$ if $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle) = \mathrm{lt}(\langle \sigma_{\mathfrak{p}'}(I) \rangle)$ has only finitely many equivalence classes and that every equivalence class is a constructible set. However there are reasons which militate against the obvious approach to simply stratify the base space $\mathrm{Spec}(A)$ with respect to the leading terms:

- The equivalence classes are indeed only constructible and not in general locally closed (see example 2).
- Even if an equivalence class Y is locally closed φ may not be flat over Y.
- Just because the function $\mathfrak{p} \mapsto \mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ is constant on Y does not mean that the reduced Gröbner bases of the fibres depend on $\mathfrak{p} \in Y$ in a "continuous way".

The following simple example illustrates the two latter points.

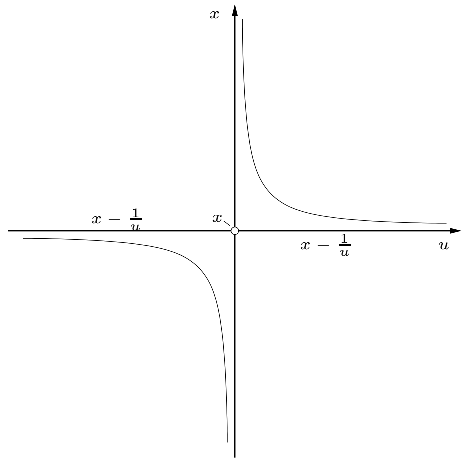
Example 3. Let k be a field and $A = k[u]$ the polynomial ring in one parameter u. Consider the ideal $I = \langle u(ux - 1), (ux - 1)x \rangle \subset A[x] = k[u, x]$. The corresponding picture is on the next page.

Geometrically the map $\varphi : \mathrm{Spec}(k[u, x]/I) \to \mathrm{Spec}(A) = \mathbb{A}^1_k$ is the projection onto the u-axis. For every point $\mathfrak{p} \in \mathrm{Spec}(A)$ the leading terms of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ are generated by x but φ is not flat: Let $\mathfrak{P} \in \mathrm{Spec}(k[u, x]/I)$ be the point corresponding to the origin in $\mathbb{A}^2_k$ then

$$\mathcal{O}_{\mathrm{Spec}(A[x]/I), \mathfrak{P}} = (A[x]/I)_{\mathfrak{P}} = k$$

because $ux - 1$ does not lie in $\mathfrak{P}$. For $\mathfrak{p} = \varphi(\mathfrak{P}) = \langle u \rangle$ we have $\mathcal{O}_{\mathrm{Spec}(A), \mathfrak{p}} = k[u]_{\langle u \rangle}$. The map $k[u]_{\langle u \rangle} \to k$ induced by φ is given by evaluation at the origin and is not flat. Thus φ is not flat at $\mathfrak{P}$.

This example suggests that the above described problems may not appear in the projective setting. Indeed we will see in section 3 that for homogeneous ideals the situation is as nice as could be hoped for, i.e. the sets over which $\mathfrak{p} \mapsto \mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ is constant are parametric.



1 Parametric sets

The idea of "parameterizing Gröbner bases" can nicely be captured using sheaves. For every subscheme Y of $\mathrm{Spec}(A)$ we will define a quasi-coherent sheaf $\mathcal{I}_Y$ on Y, which intuitively might be thought of as the restriction of I to Y.

Let Y be a locally closed subset of $\mathrm{Spec}(A)$ and $\mathfrak{a} \subset A$ the radical ideal such that $\overline{Y} = \mathrm{V}(\mathfrak{a})$ and let $\overline{I}$ denote the extension of I in $(A/\mathfrak{a})[x]$. We define $\mathcal{I}_Y$ to be the restriction of the quasi-coherent sheaf associated to the $A/\mathfrak{a}$ -module $\overline{I}$ on $\mathrm{Spec}(A/\mathfrak{a}) = \mathrm{V}(\mathfrak{a})$ to Y. That is

$$\mathcal{I}_Y = \widetilde{\overline{I}}\Big|_Y .$$

More explicitly, for an open subset U of Y the $\mathcal{O}_Y(U)$-module $\mathcal{I}_Y(U)$ consists of all functions g from U into the disjoint union $\coprod_{\mathfrak{p} \in U} \overline{I}_{\mathfrak{p}}$ which are locally fractions, i.e. for every $\mathfrak{p} \in U$ there exists an open neighborhood U' of $\mathfrak{p}$ in U such that for all $\mathfrak{q} \in U'$ we have $g(\mathfrak{q}) = \frac{P}{s} \in \overline{I}_{\mathfrak{q}}$, where $P \in \overline{I}$ and $s \in (A/\mathfrak{a}) \setminus \mathfrak{q}$ for all $\mathfrak{q} \in U'$.

Since A is noetherian, $\mathrm{Spec}(A)$ is a noetherian topological space and thus every open subset U of Y is quasi-compact. This implies that we can consider $\mathcal{I}_Y(U)$ as an ideal of the polynomial ring $\mathcal{O}_Y(U)[x]$. (If U was not quasi-compact we could not be sure that an element of $\mathcal{I}_Y(U)$ has finite support.)

Note that for $\mathfrak{p} \in Y$ the stalk $\mathcal{I}_{Y,\mathfrak{p}} = \overline{I}_{\mathfrak{p}}$ is just the extension of I under $A[x] \to (A/\mathfrak{a})_{\mathfrak{p}}[x]$. Let $\mathfrak{m}_{\mathfrak{p}}$ denote the unique maximal ideal of $\mathcal{O}_{Y,\mathfrak{p}} = (A/\mathfrak{a})_{\mathfrak{p}}$, then in analogy to the sequence

$$A \to \mathcal{O}_Y(Y) \to \mathcal{O}_{Y,\mathfrak{p}} \to \mathcal{O}_{Y,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} = k(\mathfrak{p})$$

of natural maps we obtain natural maps

$$I \to \mathcal{I}_Y(Y) \to \mathcal{I}_{Y,\mathfrak{p}} \to \langle \sigma_{\mathfrak{p}}(I) \rangle .$$

For $g \in \mathcal{I}_Y(Y)$ the image of g in $\langle \sigma_{\mathfrak{p}}(I) \rangle$ is denoted by $\overline{g}^{\mathfrak{p}}$.

Now we are prepared to give precise meaning to the intuitive idea of parameterizing Gröbner bases: We are looking for subschemes Y of $\mathrm{Spec}(A)$ with

the property that there exist global sections $g_1, \dots, g_m \in \mathcal{I}_Y(Y)$ such that for all $\mathfrak{p} \in Y$ their images $\overline{g_1}^{\mathfrak{p}}, \dots, \overline{g_m}^{\mathfrak{p}}$ form the unique reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$. We will need the following easy lemma.

Lemma 1. *Let Y be a subscheme of* $\operatorname{Spec}(A)$ *and $g, f \in \mathcal{I}_Y(Y)$. Then the set*

$$\left\{ \mathfrak{p} \in Y;\ \overline{g}^{\mathfrak{p}} = \overline{f}^{\mathfrak{p}} \right\}$$

is a closed subset of Y and $\overline{g}^{\mathfrak{p}} = \overline{f}^{\mathfrak{p}}$ for all $\mathfrak{p} \in Y$ implies $g = f$.

Proof: It suffices to treat the case $f = 0$. We can cover Y with open sets U_i such that

$$g(\mathfrak{p}) = \frac{P}{s} \in \overline{I}_{\mathfrak{p}}$$

for $P \in \overline{I} \subset (A/\mathfrak{a})[x]$ and $s \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ for all $\mathfrak{p} \in U_i$. We have

$$\{\mathfrak{p} \in Y;\ \overline{g}^{\mathfrak{p}} = 0\} \cap U_i = \{\mathfrak{p} \in U_i;\ \operatorname{coef}(P, t) \in \mathfrak{p} \text{ for all } t \in \operatorname{supp}(P)\},$$

which is a closed subset of U_i. Hence $\{\mathfrak{p} \in Y;\ \overline{g}^{\mathfrak{p}} = 0\}$ is closed.

If we interpret g as a polynomial with coefficients c_t in $\mathcal{O}_Y(Y)$, then $\overline{g}^{\mathfrak{p}} = 0$ is equivalent to saying that for all $t \in \mathcal{T}$ the image of c_t in the stalk $\mathcal{O}_{Y,\mathfrak{p}} = (A/\mathfrak{a})_{\mathfrak{p}}$ lies in the maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ of $\mathcal{O}_{Y,\mathfrak{p}}$. Since this holds for all $\mathfrak{p} \in Y$ and Y is a reduced scheme we obtain $c_t = 0 \in \mathcal{O}_Y(Y)$. Hence $g = 0$. □

Theorem 1. *If Y is a connected subscheme of* $\operatorname{Spec}(A)$ *and there exists a finite subset G of $\mathcal{I}_Y(Y)$ such that for all $\mathfrak{p} \in Y$ the set $\overline{G}^{\mathfrak{p}} = \{\overline{g}^{\mathfrak{p}};\ g \in G\}$ is the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$, then G is uniquely determined and for each $g \in G$ the function $\mathfrak{p} \mapsto \operatorname{lt}(\overline{g}^{\mathfrak{p}})$ is constant on Y. In particular, the function $\mathfrak{p} \mapsto \operatorname{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ is constant on Y.*

Proof: First we will show that for $g \in G$ and $t \in \mathcal{T}$ the set

$$W(t) = \{\mathfrak{p} \in Y;\ \operatorname{lt}(\overline{g}^{\mathfrak{p}}) = t\}$$

is a closed subset of Y. We can cover Y with open sets U_i such that

$$g(\mathfrak{p}) = \frac{P}{s} \in \overline{I}_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U_i.$$

Here $P \in \overline{I} \subset (A/\mathfrak{a})[x]$ and $s \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ for all $\mathfrak{p} \in U_i$.

Let $\mathfrak{p} \in Y$ and $\phi : (A/\mathfrak{a})_{\mathfrak{p}} \to (A/\mathfrak{a})_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} = k(\mathfrak{p})$ the canonical map. We will need that $\phi(c/s) = 1$ implies $c - s \in \mathfrak{p}$ for $c \in A/\mathfrak{a}$ and $s \in (A/\mathfrak{a}) \setminus \mathfrak{p}$. But $\phi(c/s) = 1$ is equivalent to saying that there exists $c' \in \mathfrak{p}$ and $s' \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ such that

$$\frac{c}{s} = 1 + \frac{c'}{s'} = \frac{s' + c'}{s'}.$$

This implies the existence of an $s'' \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ such that

$$(cs' - s(s' + c'))s'' = 0 \in \mathfrak{p}.$$

Hence $cs' - ss' \in \mathfrak{p}$ and therefore $c - s \in \mathfrak{p}$.

Using the above result we see that for $\mathfrak{p} \in U_i$ we have $\operatorname{lt}(\overline{g}^{\mathfrak{p}}) = t$ if and only if $\mathfrak{p}$ contains $\operatorname{coef}(P, t')$ for $t' > t$ and $\operatorname{coef}(P, t) - s$ (Use that $\overline{g}^{\mathfrak{p}}$ is monic). Therefore $W(t) \cap U_i$ is a closed subset of U_i and thus $W(t) \subset Y$ is closed.

Since $\mathrm{Spec}(A)$ is a noetherian topological space, a finite number of the U_i's will do and therefore the function $\mathfrak{p} \mapsto \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ takes only finitely many values on Y. Consequently Y is the disjoint union of finitely many $W(t)$'s. By the connectedness assumption on Y we can conclude that the function $\mathfrak{p} \mapsto \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ is constant on Y.

Assume that, for $F \subset \mathcal{I}_Y(Y)$, $\overline{F}^{\mathfrak{p}}$ is the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ for every $\mathfrak{p} \in Y$. Then for $f \in F$ and a chosen $\mathfrak{p} \in Y$ there exists a $g \in G$ such that $\overline{f}^{\mathfrak{p}} = \overline{g}^{\mathfrak{p}}$. Since the leading terms of $\overline{f}^{\mathfrak{p}}$ and $\overline{g}^{\mathfrak{p}}$ are independent of $\mathfrak{p}$ this implies $\mathrm{lt}(\overline{f}^{\mathfrak{p}}) = \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ for all $\mathfrak{p} \in Y$, but as $\overline{F}^{\mathfrak{p}} = \overline{G}^{\mathfrak{p}}$ is the reduced Gröbner basis we can conclude $\overline{f}^{\mathfrak{p}} = \overline{g}^{\mathfrak{p}}$ for all $\mathfrak{p} \in Y$ and therefore $f = g \in G$ by lemma 1. □

The following example shows that both assertions of the above theorem may be false if Y is not connected.

Example 4. Let $Y = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two distinct closed points of $\mathrm{Spec}(A)$. Note that $\mathcal{O}_Y(Y)$ is just $k(\mathfrak{p}_1) \times k(\mathfrak{p}_2)$. For $j = 1, 2$ let G_j denote the reduced Gröbner bases of $\langle \sigma_{\mathfrak{p}_j}(I) \rangle$. Then for any subset G of

$$G_1 \times G_2 \subset \langle \sigma_{\mathfrak{p}_1}(I) \rangle \times \langle \sigma_{\mathfrak{p}_2}(I) \rangle = \mathcal{I}_Y(Y)$$

with the property that the projections $G \to G_i$ are surjective we have that $\overline{G}^{\mathfrak{p}}$ is the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ for every $\mathfrak{p} \in Y$.

As we wish to have a definition suitable for all (not necessarily connected) subschemes of $\mathrm{Spec}(A)$ we simply demand what we want.

Definition 1. *A locally closed subset Y of* $\mathrm{Spec}(A)$ *is called* parametric for Gröbner bases with respect to I (and $<$) *if there exists a finite subset G of $\mathcal{I}_Y(Y)$ with the following properties:*

(1) *$\overline{G}^{\mathfrak{p}}$ is the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ for every $\mathfrak{p} \in Y$.*

(2) *For each $g \in G$ the function $\mathfrak{p} \mapsto \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ is constant on Y.*

Since the ideal $I \subset A[x]$ is clear from the context we usually omit the reference to I and simply talk about parametric subschemes of $\mathrm{Spec}(A)$.

Theorem 2. *Let $Y \subset \mathrm{Spec}(A)$ be parametric and G a finite subset of $\mathcal{I}_Y(Y)$ satisfying the two conditions of the above definition. Then G is uniquely determined and the function $\mathfrak{p} \mapsto \mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ is constant on Y. Furthermore every $g \in G$ is monic with $\mathrm{lt}(g) = \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ for every $\mathfrak{p} \in Y$.*

Proof: Because of condition (2) we can repeat the uniqueness proof as in the last paragraph of the proof of theorem 1.

To show that every $g \in G$ is monic with $\mathrm{lt}(g) = \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ observe that the coefficients of g are just elements of $\mathcal{O}_Y(Y)$. Since $(Y, \mathcal{O}_Y)$ is a reduced scheme every element of $\mathcal{O}_Y(Y)$ is uniquely determined by its images in $k(\mathfrak{p})$ where $\mathfrak{p}$ ranges over all of Y. □

Definition 2. *Let $Y \subset \mathrm{Spec}(A)$ be parametric, then the uniquely determined subset $G = G_Y$ of $\mathcal{I}_Y(Y)$ of the above theorem is called the* reduced Gröbner basis of I over Y. *We define the* leading terms of Y, *denoted by* $\mathrm{lt}(Y)$, *to be the value of the constant function* $\mathfrak{p} \mapsto \mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$.

To give the reader some idea where the journey is going we give the following definition at this early stage – even though we will not need it before section 3.

Definition 3. *A* Gröbner cover of $\operatorname{Spec}(A)$ with respect to I (and $<$) *is a finite set $\mathcal{G}$ of pairs (Y, G_Y) such that $Y \subset \operatorname{Spec}(A)$ is parametric, G_Y is the reduced Gröbner basis of I over Y and*

$$\bigcup_{(Y,G_Y)\in\mathcal{G}} Y = \operatorname{Spec}(A).$$

Parametric sets are well behaved with respect to inclusion:

Theorem 3. *Let $Y \subset \operatorname{Spec}(A)$ be parametric. Then every locally closed subset Y' of Y is parametric and the canonical map $\mathcal{I}_Y(Y) \to \mathcal{I}_{Y'}(Y')$ maps the reduced Gröbner basis of I over Y to the reduced Gröbner basis of I over Y'.*

Proof: First of all let us construct the canonical map of the theorem: Assume $\overline{Y} = \mathrm{V}(\mathfrak{a})$ and $\overline{Y'} = \mathrm{V}(\mathfrak{a}')$ for radical ideals $\mathfrak{a}$ and $\mathfrak{a}'$ of A. Let $\overline{I} \subset (A/\mathfrak{a})[x]$ and $\overline{I}' \subset (A/\mathfrak{a}')[x]$ denote the corresponding extensions of I. As $\overline{Y'} \subset \overline{Y}$ we have $\mathfrak{a} \subset \mathfrak{a}'$ and a canonical map $A/\mathfrak{a} \to A/\mathfrak{a}'$ which extends to $\varphi : \overline{I} \to \overline{I}'$. Then for $\mathfrak{p} \in Y' \subset Y$ we have a canonical map

$$\varphi_{\mathfrak{p}} : \overline{I}_{\mathfrak{p}} \to \overline{I}'_{\mathfrak{p}}.$$

Now an element $g \in \mathcal{I}_Y(Y)$ gives rise to a function

$$g' : Y' \to \prod_{\mathfrak{p}\in Y'} \overline{I}'_{\mathfrak{p}}$$

by $g'(\mathfrak{p}) = \varphi_{\mathfrak{p}}(g(\mathfrak{p}))$. One easily verifies that the map $\mathcal{I}_Y(Y) \to \mathcal{I}_{Y'}(Y'),\ g \mapsto g'$ is well defined and a morphism. For $\mathfrak{p} \in Y'$ the commutative diagram

$$\begin{tikzcd}
\overline{I}_{\mathfrak{p}} \arrow[rr] \arrow[dr] & & \overline{I}'_{\mathfrak{p}} \arrow[dl] \\
& \langle \sigma_{\mathfrak{p}}(I) \rangle &
\end{tikzcd}$$

gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{I}_Y(Y) \arrow[rr] \arrow[dr] & & \mathcal{I}_{Y'}(Y') \arrow[dl] \\
& \langle \sigma_{\mathfrak{p}}(I) \rangle &
\end{tikzcd}$$

From this the claim of the theorem follows. □

Next we will give a characterization of parametric sets in terms of monic ideals (see [19]).

Definition 4. *An ideal $I \subset A[x]$ is called* monic (with respect to $<$) *if* $\mathrm{lc}(I,t) \in \{\langle 0\rangle, \langle 1\rangle\}$ *for all $t \in \mathcal{T}$. In other words: I is monic if for every $t \in \mathrm{lt}(I)$ there exists a monic polynomial $P \in I$ with* $\mathrm{lt}(P) = t$.

There are quite a few definitions of reduced Gröbner bases in the literature. We will use the one strictly paralleling the field case.

Definition 5. *A Gröbner basis $G = \{g_1, \ldots, g_m\}$ of I is called reduced if for $j = 1 \ldots, m$*

- *g_j is monic and*
- $\mathrm{supp}(g_j) \cap \mathrm{lt}(I) = \{\mathrm{lt}(g_j)\}$.

With this definition not every ideal has a reduced Gröbner basis, but as in the field case one easily shows that if it exists, it is unique and that $A[x]/I$ is a free A-module with basis $\mathcal{T} \smallsetminus \mathrm{lt}(I)$. Concerning existence we have the following (cf. [19] and [3], theorem 2.11).

Theorem 4. *Let $I \subset A[x]$ be an ideal, then there exists a reduced Gröbner basis of I if and only if I is monic.*

Proof: If there exists a reduced Gröbner basis of I then clearly I is monic. Conversely if I is monic then we can choose monic polynomials $g_1, \ldots, g_m \in I$ such that $\mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_m)$ is the unique minimal generating set of $\mathrm{lt}(I)$. Now if we mutually reduce the g_j's we end up with the desired reduced Gröbner basis of I. $\square$

The connection to parametric subschemes is the following:

Theorem 5. *A subscheme Y of* $\mathrm{Spec}(A)$ *is parametric if and only if $\mathcal{I}_Y(Y) \subset \mathcal{O}_Y(Y)[x]$ is monic, and in this case the reduced Gröbner basis of I over Y is the reduced Gröbner basis of $\mathcal{I}_Y(Y)$. In particular* $\mathrm{lt}(\mathcal{I}_Y(Y)) = \mathrm{lt}(Y)$.

Proof: Suppose that Y is parametric and let $G \subset \mathcal{I}_Y(Y)$ denote the reduced Gröbner basis of I over Y. We will show that the leading term of every $f \in \mathcal{I}_Y(Y)$ is divisible by $\mathrm{lt}(g)$ for some $g \in G$. Since $(Y, \mathcal{O}_Y)$ is a reduced scheme there exists a $\mathfrak{p} \in Y$ such that the image of $\mathrm{lc}(f) \in \mathcal{O}_Y(Y)$ in $k(\mathfrak{p})$ is nonzero. For such a $\mathfrak{p}$ we know that $\mathrm{lt}(f) = \mathrm{lt}(\overline{f}^{\mathfrak{p}})$ is divisible by $\mathrm{lt}(\overline{g}^{\mathfrak{p}}) = \mathrm{lt}(g)$ for some $g \in G$. Since the elements of G are monic this shows that $\mathcal{I}_Y(Y)$ is monic.

Now suppose that $\mathcal{I}_Y(Y)$ is monic and let $G = \{g_1, \ldots, g_m\}$ denote the reduced Gröbner basis of $\mathcal{I}_Y(Y)$. For $f \in \mathcal{I}_Y(Y)$ the usual division (or reduction) algorithm shows that there exists a representation

$$f = f_1 g_1 + \cdots + f_m g_m$$

such that for $i = 1, \ldots, m$ we have $\mathrm{lt}(f_i)\,\mathrm{lt}(g_i) \leq \mathrm{lt}(f)$ and

$$\mathrm{coef}(f_i, t) \in \left\langle \mathrm{coef}(f, t');\ t' \geq t\,\mathrm{lt}(g_i) \right\rangle \text{ for all } t \in \mathcal{T}.$$

By the last condition we have $\mathrm{lt}(\overline{f_i}^{\mathfrak{p}})\,\mathrm{lt}(\overline{g_i}^{\mathfrak{p}}) \leq \mathrm{lt}(\overline{f}^{\mathfrak{p}})$ for every $\mathfrak{p} \in Y$. Because $\overline{f}^{\mathfrak{p}} = \overline{f_1}^{\mathfrak{p}}\overline{g_1}^{\mathfrak{p}} + \cdots + \overline{f_m}^{\mathfrak{p}}\overline{g_m}^{\mathfrak{p}}$ this shows that $\mathrm{lt}(\overline{f}^{\mathfrak{p}})$ is divisible by $\mathrm{lt}(\overline{g_i}^{\mathfrak{p}})$ for some $i \in \{1, \ldots, m\}$. Since every element of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ is of the form $\lambda \overline{f}^{\mathfrak{p}}$ for $\lambda \in k(\mathfrak{p})$

and $f \in \mathcal{I}_Y(Y)$ we can conclude that $\overline{G}^{\mathfrak{p}}$ is a Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ for every $\mathfrak{p} \in Y$. As $g \in G$ is monic the function $\mathfrak{p} \mapsto \mathrm{lt}(\overline{g}^{\mathfrak{p}})$ is clearly constant and since G is reduced also $\overline{G}^{\mathfrak{p}}$ is reduced. Thus we have shown that Y is parametric and that G is the reduced Gröbner basis of I over Y. $\square$

So the reduced Gröbner basis G of I over Y is actually a Göbner basis. In fact, by theorem 3, $G|_U = \{g|_U;\ g \in G\}$ is the reduced Gröbner basis of $\mathcal{I}_Y(U) \subset \mathcal{O}_Y(U)[x]$ for every open subset U of Y.

Corollary 1. $\mathrm{Spec}(A)$ *is parametric with respect to I if and only if I is monic and in this case the reduced Gröbner basis of I over* $\mathrm{Spec}(A)$ *is the reduced Gröbner basis of I.*

Proof: This follows directly from the theorem because $\mathcal{I}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) = I$ (see [13], chapter II, proposition 5.1). $\square$

Next we will prove a local criterion for a locally closed subset of $\mathrm{Spec}(A)$ to be parametric. Using this criterion we will then show that a family of affine or projective schemes over a parametric subset of $\mathrm{Spec}(A)$ is flat. We need two easy lemmas.

Lemma 2. *Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and $f \in I_{\mathfrak{p}}$. Then there exists $P \in I$ and $s \in A \setminus \mathfrak{p}$ such that*

$$f = \frac{P}{s} \in I_{\mathfrak{p}}$$

and $\mathrm{coef}(P, t) = 0$ *whenever* $\mathrm{coef}(f, t) = 0$. *In particular* $\mathrm{lt}(P) = \mathrm{lt}(f)$.

Proof: By definition there exists $P \in I$ and $s \in A \setminus \mathfrak{p}$ such that $f = P/s \in I_{\mathfrak{p}}$. If $\mathrm{coef}(f, t) = \mathrm{coef}(P, t)/s \in A_{\mathfrak{p}}$ is zero there exists an $s_t \in A \setminus \mathfrak{p}$ such that $\mathrm{coef}(P, t)s_t = 0$. If we multiply P and s by the product of all s_t's where t ranges over the support of P we obtain the desired representation of f. $\square$

Lemma 3. *Let $Y \subset \mathrm{Spec}(A)$ be locally closed and $\mathfrak{a} \subset A$ the radical ideal such that $\overline{Y} = \mathrm{V}(\mathfrak{a})$. Let $P \in \overline{I} \subset (A/\mathfrak{a})[x]$. Then the leading term of the image of P in $\mathcal{I}_Y(Y)$ equals the leading term of P.*

Proof: It suffices to show that there exists a $\mathfrak{p} \in Y$ which does not contain $\mathrm{lc}(P)$. Assume the contrary, then Y is contained in the closed set

$$W = \{\mathfrak{p} \in \mathrm{Spec}(A/\mathfrak{a});\ \mathrm{lc}(P) \in \mathfrak{p}\}.$$

But as Y is dense in $\mathrm{V}(\mathfrak{a}) = \mathrm{Spec}(A/\mathfrak{a})$ we conclude that $W = \mathrm{Spec}(A/\mathfrak{a})$ and thus $\mathrm{lc}(P) \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A/\mathfrak{a})$. Because $\mathfrak{a}$ is a radical ideal this yields the contradiction $\mathrm{lc}(P) = 0$.

Theorem 6. *Let $Y \subset \mathrm{Spec}(A)$ be locally closed and T' a set of terms such that $\mathcal{T}T' = T'$. Let $\mathfrak{a} \subset A$ denote the radical ideal such that $\overline{Y} = V(\mathfrak{a})$ and $\overline{I}$ the extension of I in $(A/\mathfrak{a})[x]$. Then Y is parametric with* $\mathrm{lt}(Y) = T'$ *if and only if $\overline{I}_{\mathfrak{p}}$ is monic with* $\mathrm{lt}(\overline{I}_{\mathfrak{p}}) = T'$ *for every $\mathfrak{p} \in Y$.*

Proof: To show that $\overline{I}_{\mathfrak{p}}$ is monic with $\mathrm{lt}(\overline{I}_{\mathfrak{p}}) = T'$ it suffices to prove $\mathrm{lt}(\overline{I}_{\mathfrak{p}}) \subset T'$ because this shows that the image of the reduced Gröbner basis of I over Y in $\overline{I}_{\mathfrak{p}}$ is the reduced Gröbner basis of $\overline{I}_{\mathfrak{p}}$. Let $P \in \overline{I}$ and $s \in (A/\mathfrak{a}) \setminus \mathfrak{p}$. By lemma 2 we may assume that the leading term of $P/s \in \overline{I}_{\mathfrak{p}}$ equals the leading term of

P. And by lemma 3 the leading term of the image of P in $\mathcal{I}_Y(Y)$ is the leading term of P. Thus $\mathrm{lt}(P/s) \in \mathrm{lt}(\mathcal{I}_Y(Y)) = \mathrm{lt}(Y) = T'$.

For the converse direction let $T = \{t_1, \dots, t_m\}$ denote the minimal generating set of T'. For $i = 1, \dots, m$ and $\mathfrak{p} \in Y$ let $g_i(\mathfrak{p})$ denote the element of the reduced Gröbner basis of $\overline{I}_{\mathfrak{p}}$ with leading term t_i. We want to show that g_i defines an element of $\mathcal{I}_Y(Y)$. Let $\mathfrak{p} \in Y$ and $P \in \overline{I}, s \in (A/\mathfrak{a}) \smallsetminus \mathfrak{p}$ such that $g_i(\mathfrak{p}) = P/s \in \overline{I}_{\mathfrak{p}}$. By lemma 2 we may assume $\mathrm{lt}(P) = t_i$ and $\mathrm{coef}(P, t) = 0$ for $t \in T' \smallsetminus \{t_i\}$. Because $g_i(\mathfrak{p})$ is monic there exists an $s' \in (A/\mathfrak{a}) \smallsetminus \mathfrak{p}$ such that $(\mathrm{lc}(P) - s)s' = 0$. The set $U = \{\mathfrak{q} \in Y; s, s' \notin \mathfrak{q}\}$ is an open neighborhood of $\mathfrak{p}$ in Y and we have $g_i(\mathfrak{q}) = P/s \in \overline{I}_{\mathfrak{q}}$ for all $\mathfrak{q} \in U$ because $P/s \in \overline{I}_{\mathfrak{q}}$ is monic with leading term t_i and $\mathrm{supp}(P/s) \cap T' = \{t_i\}$. Thus the g_i's are elements of $\mathcal{I}_Y(Y)$.

For $f \in \mathcal{I}_Y(Y)$ there exists a $\mathfrak{p} \in Y$ such that the image of $\mathrm{lc}(f)$ in $(A/\mathfrak{a})_{\mathfrak{p}}$ is nonzero. This implies that the leading term of the image of f in $\overline{I}_{\mathfrak{p}}$ is the leading term of f and thus we have $\mathrm{lt}(f) \in \mathrm{lt}(\overline{I}_{\mathfrak{p}}) = T'$.

Consequently $\mathrm{lt}(\mathcal{I}_Y(Y)) = T'$ and because g_i is monic with leading term t_i for $i = 1, \dots, m$ by theorem 5 we see that Y is parametric. □

Recall that φ denotes the map from $\mathrm{Spec}(A[x]/I)$ respectively $\mathrm{Proj}(A[x]/I)$ to $\mathrm{Spec}(A)$.

Corollary 2. *If $Y \subset \mathrm{Spec}(A)$ is parametric then φ is flat over Y, i.e. the map $\varphi^{-1}(Y) \to Y$ is a flat morphism.*

Proof: Let $\mathfrak{a} \subset A$ denote the radical ideal such that $\overline{Y} = \mathrm{V}(\mathfrak{a})$ and let $\overline{I}$ denote the extension of I in $(A/\mathfrak{a})[x]$. The scheme structure on the set $\varphi^{-1}(Y)$ is given by identifying $\varphi^{-1}(Y)$ with $X = \mathrm{Spec}(A[x]/I) \times_A Y$ or $X = \mathrm{Proj}(A[x]/I) \times_A Y$ respectively. Thus for $\mathfrak{P} \in \varphi^{-1}(Y)$ the stalk $\mathcal{O}_{X,\mathfrak{P}}$ equals $((A/\mathfrak{a})[x]/\overline{I})_{\mathfrak{P}}$ or $((A/\mathfrak{a})[x]/\overline{I})_{(\mathfrak{P})}$. (Here $((A/\mathfrak{a})[x]/\overline{I})_{(\mathfrak{P})}$ denotes the elements of degree zero in the localized ring $S^{-1}((A/\mathfrak{a})[x]/\overline{I})$, where S is the multiplicative system of all homogeneous elements of $(A/\mathfrak{a})[x]/\overline{I}$ which do not lie in $\mathfrak{P}$.) Let $\mathfrak{p} = \varphi(\mathfrak{P}) \in Y$. We have to show that $\varphi_{\mathfrak{P}} : \mathcal{O}_{Y,\mathfrak{p}} \to \mathcal{O}_{X,\mathfrak{P}}$ is flat. In the affine case $\varphi_{\mathfrak{P}}$ can be factored:

$$\mathcal{O}_{Y,\mathfrak{p}} = (A/\mathfrak{a})_{\mathfrak{p}} \to (A/\mathfrak{a})_{\mathfrak{p}}[x]/\overline{I}_{\mathfrak{p}} = ((A/\mathfrak{a})[x]/\overline{I})_{\mathfrak{p}} \to ((A/\mathfrak{a})[x]/\overline{I})_{\mathfrak{P}} = \mathcal{O}_{X,\mathfrak{P}}.$$

By theorem 6 the ideal $\overline{I}_{\mathfrak{p}} \subset (A/\mathfrak{a})_{\mathfrak{p}}[x]$ is monic and thus $(A/\mathfrak{a})_{\mathfrak{p}}[x]/\overline{I}_{\mathfrak{p}}$ is a free $(A/\mathfrak{a})_{\mathfrak{p}}$-module. In particular $(A/\mathfrak{a})_{\mathfrak{p}}[x]/\overline{I}_{\mathfrak{p}}$ is a flat $(A/\mathfrak{a})_{\mathfrak{p}}$-module. Since "localization is flat" $((A/\mathfrak{a})[x]/\overline{I})_{\mathfrak{P}}$ is a flat $((A/\mathfrak{a})[x]/\overline{I})_{\mathfrak{p}}$-module. This completes the proof in the affine case.

In the projective case we know that $S^{-1}((A/\mathfrak{a})[x]/\overline{I})$ is a flat $((A/\mathfrak{a})[x]/\overline{I})_{\mathfrak{p}}$-module and therefore also a flat $(A/\mathfrak{a})_{\mathfrak{p}}$-module. Since $((A/\mathfrak{a})[x]/\overline{I})_{(\mathfrak{P})}$ is a direct summand of $S^{-1}((A/\mathfrak{a})[x]/\overline{I})$ also $((A/\mathfrak{a})[x]/\overline{I})_{(\mathfrak{P})}$ is a flat $(A/\mathfrak{a})_{\mathfrak{p}}$-module. □

2 Lucky primes and pseudo division

Now it is time to introduce the concept of pseudo division (cf. [6] and [17]). This is basically just the usual division without fractions. The idea behind pseudo division already appeared in the proof of theorem 5.

Definition 6. *Let* $f, g_1, \ldots, g_m \in A[x]$. *A representation*

$$cf = f_1 g_1 + \cdots + f_m g_m + r$$

is called a pseudo division of f modulo $g_1, \ldots, g_m$ (w.r.t. $<$) *if the following assertions are satisfied:*

- $f_1, \ldots, f_m, r \in A[x]$ *and* $c \in A$ *is a product of leading coefficients of the* g_j*'s.*
- $\operatorname{lt}(f_j) \operatorname{lt}(g_j) \leq \operatorname{lt}(f)$ *for* $j = 1, \ldots, m$.
- *No term in* $\operatorname{supp}(r)$ *is divisible by a leading term of the* g_j*'s.*
- $\operatorname{coef}(f_j, t) \in \left\langle \operatorname{coef}(f, t');\ t' \geq \operatorname{lt}(g_j)t \right\rangle$ *for all* $j \in \{1, \ldots, m\}$ *and* $t \in \mathcal{T}$.

In this situation r is called a remainder of f after pseudo division modulo $g_1, \ldots, g_m$. A pseudo division of f modulo $g_1, \ldots, g_m$ can be obtained by successively applying pseudo reduction steps:

If there exists an element of the support of f which is divisible by a leading term of any of the g_j's then choose $t \in \operatorname{supp}(f)$ maximal with this property. Then $t = t' \operatorname{lt}(g_j)$ holds for some $j \in \{1, \ldots, m\}$ and $t' \in \mathcal{T}$. Now substitute f by

$$\operatorname{lc}(g_j) f - \operatorname{coef}(f, t) t' g_j.$$

By iterating this process and keeping track of the monomials used, we obtain the desired representation.

The nice thing about pseudo reductions is that they are stable under specialization in the sense that

$$\operatorname{lt}(\overline{f_j}) \operatorname{lt}(\overline{g_j}) \leq \operatorname{lt}(\overline{f})$$

for $j = 1, \ldots, m$. Here $\overline{g}$ denotes the coefficientwise reduction of $g \in A[x]$ modulo some ideal of A. (This follows directly from the last assertion of the definition.)

Observe that c may well be zero if A is not an integral domain.

Definition 7. *A prime ideal of* A *is called lucky for* I *if for every* $t \in \operatorname{lt}(I)$ *it does not contain* $\operatorname{lc}(I, t)$.

To my knowledge the expression "lucky" was coined by mathematicians working on modular algorithms to compute Gröbner bases over $\mathbb{Q}$ (see [2], [19], [11]). Mod p-arithmetic avoids the phenomenon of coefficient growth but it is not a priori clear which prime numbers p can be used for lifting a Gröbner basis over $\mathbb{Z}/\mathbb{Z}p$ to a Gröbner basis over $\mathbb{Q}$. So mathematicians must have considered themselves lucky when they picked a prime doing the job.

Let T be the unique minimal generating set of $\operatorname{lt}(I)$. Because $\operatorname{lc}(I, t) \subset \operatorname{lc}(I, t')$ if t divides t', a prime $\mathfrak{p} \in \operatorname{Spec}(A)$ is lucky for I if and only if $\mathfrak{p}$ does not contain $\prod_{t \in T} \operatorname{lc}(I, t)$. In particular luckiness is an open condition.

Definition 8. *The ideal*

$$J = J(I) = \sqrt{\prod_{t \in T} \operatorname{lc}(I, t)} \subset A$$

is called the singular ideal of I (with respect to $<$).

So a prime $\mathfrak{p} \in \mathrm{Spec}(A)$ is unlucky (i.e. not lucky) for I if and only if it is an element of the singular variety $\mathrm{V}(J)$.

In [22] Weispfenning introduced another discriminant ideal which, however, can only be constructed if A is an integral domain. So for the time being assume that A is an integral domain. In this case we can consider the reduced Gröbner basis G of I over the quotient field of A. For $g \in G$ the set

$$J_g = \{a \in A;\ ag \in I\}$$

clearly is an ideal of A and we can define Weispfenning's discriminant ideal by

$$J' = J'(I) = \sqrt{\prod_{g \in G} J_g}.$$

Clearly $J_g \subset \mathrm{lc}(I, \mathrm{lt}(g))$ always holds but the inclusion may be strict as illustrated by the following example.

Example 5. Let k be a field and $A = k[u_1, u_2]$ the polynomial ring in the parameters u_1, u_2. We consider the ideal

$$I = \langle u_1 x + u_2,\ u_1 y^2 - 1 \rangle \subset A[x, y].$$

With respect to any term order the reduced Gröbner basis of I over the quotient field of A is

$$G = \left\{ x + \frac{u_2}{u_1},\ y^2 - \frac{1}{u_1} \right\}.$$

But as $u_2 y^2 + x = y^2(u_1 x + u_2) - x(u_1 y^2 - 1) \in I$ we have with respect to any term order with $y^2 > x$

$$J_{y^2 - \frac{1}{u_1}} = \langle u_1 \rangle \subsetneqq \langle u_1, u_2 \rangle \subset \mathrm{lc}(I, y^2).$$

However our discriminant ideal is not larger than Weispfenning's; in fact, they are the same.

Theorem 7. *In the above described situation we have $J = J'$.*

Proof: Let I' denote the extension of I in the polynomial ring over the quotient field of A. First of all observe that $\mathrm{lt}(I) = \mathrm{lt}(I')$: As $I \subset I'$ the inclusion $\mathrm{lt}(I) \subset \mathrm{lt}(I')$ is clear. For the other inclusion it suffices to notice that every $P \in I'$ is of the form $P = \frac{Q}{a}$ with $Q \in I$ and $a \in A$.

Let $G = \{g_1, \ldots, g_m\}$ denote the unique reduced Gröbner basis of I' over the quotient field of A. Then as $\mathrm{lt}(I) = \mathrm{lt}(I')$ the unique minimal generating set T of $\mathrm{lt}(I)$ equals $\{\mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_m)\}$. With the abbreviations $t_j = \mathrm{lt}(g_j)$ and $J_j = J_{g_j}$ for $j = 1, \ldots, m$ we may assume $t_1 < \cdots < t_m$. We have to show

$$\mathrm{V}\left(\mathrm{lc}(I, t_1) \cdots \mathrm{lc}(I, t_m)\right) = \mathrm{V}(J_1 \cdots J_m).$$

As $J_j \subset \mathrm{lc}(I, t_j)$ for $j = 1, \ldots, m$, the inclusion " $\subset$ " is clear. For the other inclusion it will suffice to show that for $j \in \{1, \ldots, m\}$ and $\mathfrak{p} \in \mathrm{Spec}(A)$

$$J_j \subset \mathfrak{p} \ \Rightarrow\ \mathrm{lc}(I, t_1) \cdots \mathrm{lc}(I, t_j) \subset \mathfrak{p}.$$

We will prove this by contradiction. So assume $\mathrm{lc}(I, t_1) \cdots \mathrm{lc}(I, t_j) \not\subseteq \mathfrak{p}$. Then we can find $f_1, \ldots, f_j \in I$ with $\mathrm{lt}(f_i) = t_i$ and $\mathrm{lc}(f_i) \notin \mathfrak{p}$ for $i = 1, \ldots, j$. Pseudo reduction of f_j modulo $f_1, \ldots, f_{j-1}$ yields a polynomial $g \in I$ with $\mathrm{lt}(g) = t_j$, $\mathrm{lc}(g) \notin \mathfrak{p}$ and no term in $\mathrm{supp}(g)$ divisible by any $t_1, \ldots, t_{j-1}$. So no term in the support of $g - \mathrm{lc}(g) g_j \in I'$ is divisible by any $t_1, \ldots, t_m$. Hence $\mathrm{lc}(g) g_j = g \in I$ and we conclude $\mathrm{lc}(g) \in J_j \subset \mathfrak{p}$ (in contradiction to $\mathrm{lc}(g) \notin \mathfrak{p}$). □

The above theorem asserts that the concept of (in)essential specializations as introduced by Weispfenning in [22] is equivalent to the older concept of (un)lucky prime ideals. The advantage of the idea of luckiness is, of course, that it works for more general rings, i.e. not only for integral domains. Observe that it is quite natural to work with rings which are not integral domains, because even if you start with an integral domain (e.g. the polynomial ring over a field in some parameters), the singular ideal J will typically not be prime and therefore A/J will not be an integral domain. The relevance of this will become clear with the next theorem which gives a characterization of parametric subsets in terms of luckiness.

Lemma 4. *Let $Y \subset \mathrm{Spec}(A)$ be parametric and $\mathfrak{a} \subset A$ the radical ideal such that $\overline{Y} = \mathrm{V}(\mathfrak{a})$. If $\overline{I}$ denotes the extension of I in $(A/\mathfrak{a})[x]$ then $\mathrm{lt}(Y) = \mathrm{lt}(\overline{I})$.*

Proof: Let $t \in \mathrm{lt}(Y)$ and $\mathfrak{p} \in Y$. From theorem 6 we know that $\overline{I}_{\mathfrak{p}} \subset (A/\mathfrak{a})_{\mathfrak{p}}[x]$ is monic with $\mathrm{lt}(\overline{I}_{\mathfrak{p}}) = \mathrm{lt}(Y)$. Thus there exists $P \in \overline{I}$ and $s \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ such that the leading term of $P/s \in \overline{I}_{\mathfrak{p}}$ equals t. By lemma 2 we may assume $t = \mathrm{lt}(P) \in \mathrm{lt}(\overline{I})$.

The inclusion $\mathrm{lt}(\overline{I}) \subset \mathrm{lt}(Y)$ follows from lemma 3 and theorem 5. □

Now we are prepared to prove the main theorem of this section. It exhibits the "geometric meaning" of luckiness.

Theorem 8. *Let Y be a locally closed subset of $\mathrm{Spec}(A)$ and $\mathfrak{a} \subset A$ the radical ideal such that $\overline{Y} = \mathrm{V}(\mathfrak{a})$. Denote by $\overline{I}$ the image of I in $(A/\mathfrak{a})[x]$. Then Y is parametric for Gröbner bases with respect to I if and only if*

$$Y \cap \mathrm{V}(J(\overline{I})) = \emptyset.$$

In other words: Y is parametric if and only if every $\mathfrak{p} \in Y$ is lucky for $\overline{I}$.

Proof: Assume Y is parametric and $\{g_1, \ldots, g_m\} \subset \mathcal{I}_Y(Y)$ is the reduced Gröbner basis of I over Y. Then by lemma 4 the minimal generating set T of $\mathrm{lt}(\overline{I})$ equals $\{\mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_m)\}$. Let $\mathfrak{p} \in Y$ and $i \in \{1, \ldots, m\}$. By lemma 2 there exists $P_i \in \overline{I}$ with $\mathrm{lt}(P_i) = \mathrm{lt}(g_i(\mathfrak{p}))$ and $s_i \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ such that $g_i(\mathfrak{p}) = P_i/s_i \in \overline{I}_{\mathfrak{p}}$. Because $\mathrm{lt}(P_i) = \mathrm{lt}(g_i(\mathfrak{p})) = \mathrm{lt}(g_i) = \mathrm{lt}(\overline{g_i}^{\mathfrak{p}})$ we have $\mathrm{lc}(P_i) \notin \mathfrak{p}$, i.e. $\mathrm{lc}(\overline{I}, \mathrm{lt}(P_i)) \not\subseteq \mathfrak{p}$. Hence

$$J(\overline{I}) = \prod_{t \in T} \mathrm{lc}(\overline{I}, t) \not\subseteq \mathfrak{p}.$$

For the converse direction fix a $\mathfrak{p} \in Y$ first and let $T = \{t_1, \ldots, t_m\}$ denote the minimal generating set of $\mathrm{lt}(\overline{I})$. By assumption

$$\prod_{i=1}^{m} \mathrm{lc}(\overline{I}, t_i) \not\subseteq \mathfrak{p}.$$

Hence there exist polynomials $P_1, \dots, P_m \in \overline{I}$ with $\mathrm{lt}(P_i) = t_i$ and $\mathrm{lc}(P_i) \notin \mathfrak{p}$. For $i = 1, \dots, m$ let $Q_i \in \overline{I}$ denote a remainder of P_i after pseudo division modulo $\{P_1, \dots, P_m\} \smallsetminus \{P_i\}$. Note that $\mathrm{lt}(Q_i) = \mathrm{lt}(P_i) = t_i$ and $\mathrm{lc}(Q_i)$ is a product of leading coefficients of the P_j's. Define

$$U = \{\mathfrak{q} \in Y;\ \mathrm{lc}(P_1) \cdots \mathrm{lc}(P_m) \notin \mathfrak{q}\}.$$

Then U is an open neighborhood of $\mathfrak{p} \in Y$ and $Q_i / \mathrm{lc}(Q_i)$ defines an element of $\mathcal{I}_Y(U)$, which by abuse of notation we again denote by $Q_i / \mathrm{lc}(Q_i)$.

We can repeat the above construction for any $\mathfrak{p}' \in Y$ to obtain U' and Q_i' (analogously defined). To obtain global sections $g_i \in \mathcal{I}_Y(Y)$ we have to show that

$$\left.\frac{Q_i}{\mathrm{lc}(Q_i)}\right|_{U \cap U'} = \left.\frac{Q_i'}{\mathrm{lc}(Q_i')}\right|_{U \cap U'}.$$

The leading term of

$$\mathrm{lc}(Q_i')Q_i - \mathrm{lc}(Q_i)Q_i' \in \overline{I}$$

is strictly smaller than t_i and by construction no term in the support of $\mathrm{lc}(Q_i')Q_i - \mathrm{lc}(Q_i)Q_i'$ is divisible by an element of $\{t_1, \dots, t_m\} \smallsetminus \{t_i\}$. Thus $\mathrm{lc}(Q_i')Q_i - \mathrm{lc}(Q_i)Q_i' = 0$ and we can glue together the sections $Q_i / \mathrm{lc}(Q_i) \in \mathcal{I}_Y(U)$ to obtain global sections $g_i \in \mathcal{I}_Y(Y)$.

To show that Y is parametric we will prove that $G = \{g_1, \dots, g_m\}$ satisfies the conditions of definition 1. Clearly $\mathrm{lt}(\overline{g_i}^{\mathfrak{p}}) = t_i$ for every $\mathfrak{p} \in Y$. So it remains to show that $\overline{G}^{\mathfrak{p}}$ is the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ for every $\mathfrak{p} \in Y$. Let $\mathfrak{p} \in Y$ and $P \in \overline{I}$. For a pseudo division (see definition 6)

$$cP = P_1 Q_1 + \cdots + P_m Q_m + r$$

of P modulo $Q_1, \dots, Q_m$ we have $r \in \overline{I}$, but no term in the support of r is divisible by an element of $\{\mathrm{lt}(Q_1), \dots, \mathrm{lt}(Q_m)\} = T$. Thus $r = 0$ and

$$cP = P_1 Q_1 + \cdots + P_m Q_m.$$

Let $\phi : (A/\mathfrak{a})[x] \to k(\mathfrak{p})[x]$ denote the natural map then

$$\phi(c)\phi(P) = \phi(P_1)\phi(Q_1) + \cdots + \phi(P_m)\phi(Q_m)$$

and $\mathrm{lt}(\phi(P_i))\,\mathrm{lt}(\phi(Q_i)) \leq \mathrm{lt}(\phi(P))$. Since $\mathrm{lc}(Q_i) \notin \mathfrak{p}$ and c is a product of leading coefficients of the Q_i's we know that $\phi(c), \phi(\mathrm{lc}(Q_1)), \dots, \phi(\mathrm{lc}(Q_m))$ are all nonzero. Consequently $\mathrm{lt}(\phi(P))$ is divisible by $\mathrm{lt}(\phi(Q_i)) = t_i$ for some $i \in \{1, \dots, m\}$. Since every element of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ is of the form λf for $\lambda \in k(\mathfrak{p})$ and $f \in \phi(\overline{I}) = \sigma_{\mathfrak{p}}(I)$ this shows that $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ is generated by T and so indeed $\overline{G}^{\mathfrak{p}}$ is a Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$. Clearly $\overline{g_i}^{\mathfrak{p}}$ is monic and by construction of the Q_i's no term in the support of $\overline{g_i}^{\mathfrak{p}}$ is divisible by an element of $T \smallsetminus \{t_i\}$. Thus $\overline{G}^{\mathfrak{p}}$ is the reduced Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ and we are done. □

Definition 9. *Let Z be a closed subset of* $\mathrm{Spec}(A)$ *and $\mathfrak{a} \subset A$ the radical ideal such that $Z = \mathrm{V}(\mathfrak{a})$. Let furthermore $\overline{I}$ denote the extension of I in $(A/\mathfrak{a})[x]$. We define*

$$Z_{gen} = Z \smallsetminus \mathrm{V}(J(\overline{I})).$$

Theorem 9. *Let $Z \subset \operatorname{Spec}(A)$ be closed, $\mathfrak{a} \subset A$ the radical ideal such that $Z = \operatorname{V}(\mathfrak{a})$ and $\overline{I}$ the extension of I in $(A/\mathfrak{a})[x]$. Then Z_{gen} is parametric with $\operatorname{lt}(Z_{gen}) = \operatorname{lt}(\overline{I})$. Furthermore if Y is an open subset of Z such that Y is parametric with $\operatorname{lt}(Y) = \operatorname{lt}(\overline{I})$ then $Y \subset Z_{gen}$.*

In other words: Z_{gen} is the largest open parametric subset of Z with the same leading terms as $\overline{I}$.

Proof: Let Y be an open subset of Z. First we will show that $\mathcal{I}_Z|_Y$ is canonically isomorphic to $\mathcal{I}_Y$. Let $\mathfrak{a}' \subset A$ denote the radical ideal such that $\overline{Y} = \operatorname{V}(\mathfrak{a}')$ and $\overline{I}'$ the extension of I in $(A/\mathfrak{a}')[x]$. Then $\mathfrak{a} \subset \mathfrak{a}'$ and the canonical map $A/\mathfrak{a} \to A/\mathfrak{a}'$ extends to $\overline{I} \to \overline{I}'$ and further to $\phi : \overline{I}_\mathfrak{p} \to \overline{I}'_\mathfrak{p}$ for $\mathfrak{p} \in Y$. It suffices to show that ϕ is an isomorphism.

Clearly ϕ is surjective. Let $P \in I$ and $s \in A \setminus \mathfrak{p}$ such that $\phi(\overline{P}/s) \in \overline{I}'_\mathfrak{p}$ is zero. This means that there exists $s' \in A \setminus \mathfrak{p}$ such that $\operatorname{coef}(s'P, t) \in \mathfrak{a}'$ for every $t \in \mathcal{T}$. Let $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ be the (unique minimal) primary decomposition of the radical ideal $\mathfrak{a}$. We may assume $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in Y$ and $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_m \notin Y$. Note that $\mathfrak{p}_i \notin Y$ implies $\operatorname{V}(\mathfrak{p}_i) \cap Y = \emptyset$ because Y is an open subset of Z. So in particular $\mathfrak{p}_i \not\subseteq \mathfrak{p}$ for $i = r+1, \ldots, m$. This means that there exists an $s'' \in \mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_m \setminus \mathfrak{p}$. For $1 \le i \le r$ we have $\operatorname{V}(\mathfrak{p}_i) \subset \overline{Y} = \operatorname{V}(\mathfrak{a}')$ and thus $\mathfrak{a}' \subset \mathfrak{p}_i$. Combining these results we see that every coefficient of $s''s'P$ lies in $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m = \mathfrak{a}$ and thus $\overline{P}/s$ is zero in $\overline{I}_\mathfrak{p}$. Consequently ϕ is injective.

An argument similar to the one above shows that for $\mathfrak{p} \in Y$ the map $(A/\mathfrak{a})_\mathfrak{p} \to (A/\mathfrak{a}')_\mathfrak{p}$ is an isomorphism. Thus ϕ also preserves leading terms.

Now to show that Z_{gen} is parametric with the same leading terms as $\overline{I}$, just repeat the second part of the proof of theorem 8 (with Z_{gen} instead of Y) and use that $\mathcal{I}_Z(Z_{gen})$ is canonically isomorphic to $\mathcal{I}_{Z_{gen}}(Z_{gen})$.

Now we additionally assume that Y is a parametric subset of Z with $\operatorname{lt}(Y) = \operatorname{lt}(\overline{I})$. Suppose $Y \not\subseteq Z_{gen}$. Then there exists a $\mathfrak{p} \in Y \setminus Z_{gen}$. Let T denote the minimal generating set of $\operatorname{lt}(\overline{I})$. Since $\mathfrak{p} \notin Z_{gen} = Z \setminus \operatorname{V}(J(\overline{I}))$ there exists a $t \in T$ such that $\operatorname{lc}(\overline{I}, t) \subset \mathfrak{p}$.

Since Y is parametric with $\operatorname{lt}(Y) = \operatorname{lt}(\overline{I})$ we know from theorem 6 that $\overline{I}'_\mathfrak{p}$ is monic with $\operatorname{lt}(\overline{I}'_\mathfrak{p}) = \operatorname{lt}(\overline{I})$. Using the isomorphism $\phi : \overline{I}_\mathfrak{p} \to \overline{I}'_\mathfrak{p}$ we see that $\overline{I}_\mathfrak{p}$ is monic with $\operatorname{lt}(\overline{I}_\mathfrak{p}) = \operatorname{lt}(\overline{I})$. Thus there exists $P \in \overline{I}$ and $s \in A \setminus \mathfrak{p}$ such that $P/s \in \overline{I}_\mathfrak{p}$ is monic with leading term t. By lemma 2 we may assume $\operatorname{lt}(P) = t$. Since P/s is monic there exists $s' \in A \setminus \mathfrak{p}$ such that $(\operatorname{lc}(P) - s)s' = 0$. Thus $\operatorname{lc}(P) \notin \mathfrak{p}$ in contradiction to $\operatorname{lc}(\overline{I}, t) \subset \mathfrak{p}$. □

If we take $Z = \operatorname{Spec}(A)$ in the above theorem, then we see that the set of all lucky primes of A ($= \operatorname{Spec}(A) \setminus \operatorname{V}(J(I))$) is the largest open parametric subset of $\operatorname{Spec}(A)$ with the same leading terms as I. This more or less comes down to saying that J is the optimal discriminant ideal.

Caution: It is not true that $\mathfrak{p} \in \operatorname{Spec}(A)$ is lucky for I if and only if $\operatorname{lt}(I) = \operatorname{lt}(\langle \sigma_\mathfrak{p}(I) \rangle)$. We have seen above that the "only if" direction is correct but the "if" direction is not true in general (see example 3). However it is true for homogeneous ideals as we will see in section 3.1.

The following simple example illustrates that Z_{gen} may well be the empty set.

Example 6. Assume that A is not an integral domain, then there exist $a, b \in$

$A \setminus \{0\}$ such that $ab = 0$. If we take I to be the ideal of $A[x_1, x_2]$ generated by ax_1 and bx_2 then (with respect to any term order) $J(I) = \langle 0 \rangle$ and so $\mathrm{Spec}(A)_{gen} = \emptyset$.

However, this cannot happen if Z is irreducible, because then $Z = \mathrm{V}(\mathfrak{a})$ for some prime ideal $\mathfrak{a}$ of A and since $A/\mathfrak{a}$ is an integral domain $J(\overline{I})$ is not the zero ideal and thus Z_{gen} is nonempty. In particular Z_{gen} is dense in Z and contains the generic point of Z.

The following examples have been included to convince the reader that the singular ideal J is quite a reasonable object.

Example 7. Let $I \subset A[x]$ be the ideal generated by a square linear system

$$\begin{aligned} P_1 &= b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n - c_1 \\ &\;\;\vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ P_n &= b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n - c_n \end{aligned}$$

and let

$$B = \big(b_{ij}\big)_{1 \le i,j \le n} \in A^{n \times n}$$

denote the matrix of the system. Suppose $\det = \det(B) \in A$ is not a zero divisor. Then the singular ideal J of I is independent of the chosen term order and $\mathrm{V}(J)$ equals $\mathrm{V}(\det)$. In other words $J = \sqrt{\langle \det \rangle}$.

Proof: Let $B' \in A^{n \times n}$ denote the adjoint matrix of B. A classical linear algebra theorem (see e.g. [15], chapter 8, § 4, proposition 8) asserts that

$$B'B = BB' = \det \cdot \mathbb{1}, \tag{1}$$

where $\mathbb{1}$ denotes the $n \times n$ identity matrix.

First we show that $1 \notin \mathrm{lt}(I)$. Suppose the contrary. Let A' denote the total ring of fractions of A, i.e. the localization at the multiplicative subset of all nonzero divisors. Then we may regard A as a subring of A'. With the abbreviations

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \text{ and } \xi = \frac{1}{\det} \cdot B'c$$

identity (1) shows that ξ is a solution of our linear system. Now $1 \in \mathrm{lt}(I)$ simply means that there exist an $a \in A \setminus \{0\}$ and $Q_1, \ldots, Q_n \in A[x]$ such that

$$Q_1P_1 + \cdots + Q_nP_n = a.$$

Evaluation at ξ yields the contradiction $a = 0$.

Identity (1) also shows that det lies in $\mathrm{lc}(I, x_i)$ for $i = 1, \ldots, m$. Therefore $\det \in J$ and $\mathrm{V}(J) \subset \mathrm{V}(\det)$. Now for the converse inclusion assume $\mathfrak{p} \in \mathrm{V}(\det)$, i.e. $\det \in \mathfrak{p}$. From theorem 9 we know that for every $\mathfrak{q} \in \mathrm{Spec}(A) \setminus \mathrm{V}(J)$ the leading terms of $\langle \sigma_{\mathfrak{q}}(I) \rangle$ are generated by $x_1, \ldots, x_n$. But $\det \in \mathfrak{p}$ implies that $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ is not generated by $x_1, \ldots, x_n$ and consequently $\mathfrak{p} \in \mathrm{V}(J)$. □

Example 8. Let k be a field and $I' \subset k[x] = k[x_1, \ldots, x_n]$ a (homogeneous) ideal. For $1 \le i, j \le n$ let u_{ij} be additional indeterminates and abbreviate

$$ux = (u_{11}x_1 + \cdots + u_{n1}x_n, \ldots\ldots, u_{1n}x_1 + \cdots + u_{nn}x_n).$$

Let A be the polynomial ring over k in the u_{ij}'s and define

$$I = \langle P(ux);\ P \in I' \rangle \subset A[x].$$

Then the ideal of $k[x]$ generated by $\mathrm{lt}(\mathrm{Spec}(A)_{gen})$ is the generic initial ideal of I', usually denoted by $\mathrm{Gin}(I')$ (see e.g. [7] or [12]).

Example 9. Suppose that $<$ is a graded order and A is an integral domain, i.e. $\mathrm{Spec}(A)$ is irreducible. Then $\mathrm{Spec}(A)_{gen}$ is a nonempty, open (and thus dense) subset of $\mathrm{Spec}(A)$ such that the function

$$\mathfrak{p} \mapsto \text{affine Hilbert function of } \langle \sigma_{\mathfrak{p}}(I) \rangle$$

is constant on $\mathrm{Spec}(A)_{gen}$. This is clear because the affine Hilbert function of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ is determined by $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ (see [6], chapter 9, § 3, proposition 4). Of course there is also an analogous "projective" statement.

3 Gröbner covers

Now that we have (at least to some extent) explored the nature of parametric sets, it is time to see the complete picture.

Definition 10. *Let L be a locally closed subset of* $\mathrm{Spec}(A)$. *A finite set $\mathcal{G}$ consisting of pairs (Y, G_Y) with $Y \subset \mathrm{Spec}(A)$ parametric and G_Y the reduced Gröbner bases of I over Y is called a* Gröbner cover of L with respect to I (and $<$) *if*

$$L = \bigcup_{Y \in \mathcal{G}} Y.$$

A Gröbner cover $\mathcal{G}$ is called irreducible *if every $Y \in \mathcal{G}$ is irreducible.*

A Gröbner cover $\mathcal{G}$ of L is called locally maximal *if for every $Y \in \mathcal{G}$ the following holds: If $Y' \subset \mathrm{Spec}(A)$ is parametric with $Y' \subset L$ and $Y \subset Y' \subset \overline{Y}$ then $Y = Y'$.*

A Gröbner cover $\mathcal{G}$ is called small *if for every $Y \in \mathcal{G}$ we have*

$$\overline{Y \setminus \bigcup_{Y' \in \mathcal{G} \setminus \{Y\}} Y'} = \overline{Y}.$$

As already done in the above definition we write $Y \in \mathcal{G}$ instead of unhandy $(Y, G_Y) \in \mathcal{G}$ and refer to Y as an element of $\mathcal{G}$. To say that a Gröbner cover is small basically means that its elements are not unnecessarily large. Our main interest, of course, is in Gröbner covers of $\mathrm{Spec}(A)$ but (with a view towards applications) it seems reasonable to also treat the relative case.

Definition 11. *Let L be a locally closed subset of* $\mathrm{Spec}(A)$ *and G a finite subset of I. Then G is called a* comprehensive Gröbner basis of I with respect to L (and $<$) *if $\sigma_{\mathfrak{p}}(G) = \{\sigma_{\mathfrak{p}}(g);\ g \in G\}$ is a Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ for every $\mathfrak{p} \in L$.*

Comprehensive Gröbner bases were introduced by Weispfenning in [21] and advanced in [22]. There is a rather obvious connection between Gröbner covers

of L and comprehensive Gröbner bases of I with respect to L, which we will now describe.

Let $\mathcal{G}$ be a Gröbner cover of L. Choose a $Y \in \mathcal{G}$ and let $\mathfrak{a} \subset A$ be the radical ideal such that $\overline{Y} = \mathrm{V}(\mathfrak{a})$, furthermore let $\overline{I}$ denote the image of I in $(A/\mathfrak{a})[x]$. Since $\mathrm{Spec}(A)$ is a noetherian topological space, Y is quasi-compact and so for every $g \in G_Y$ we can find finitely many open subsets U_i of Y which cover Y and have the following property: There exists a $P \in I$ and $s \in A/\mathfrak{a}$ such that

$$g(\mathfrak{p}) = \frac{\overline{P}}{s} \in \overline{I}_{\mathfrak{p}} \text{ for every } \mathfrak{p} \in U_i.$$

Here $\overline{P}$ denotes the image of P in $\overline{I} \subset (A/\mathfrak{a})[x]$. Now taking together all such P's (for all U_i's, all $g \in G_Y$ and all $Y \in \mathcal{G}$) we end up with a finite subset of I which clearly is a comprehensive Gröbner basis of I with respect to L.

The main theorem of this section asserts that for every locally closed subset L of $\mathrm{Spec}(A)$ there exists a unique irreducible, small and locally maximal Gröbner cover of L. For the proof we will need a few basic facts about constructible sets (cf. [13]).

Definition 12. *Let X be a topological space. A constructible subset of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that*

(1) *every open subset is in $\mathfrak{F}$,*

(2) *a finite intersection of elements in $\mathfrak{F}$ is in $\mathfrak{F}$, and*

(3) *the complement of an element in $\mathfrak{F}$ is in $\mathfrak{F}$.*

One easily shows that the constructible sets of a topological space are exactly the finite unions of locally closed sets.

Lemma 5. *Let C be a constructible subset of* $\mathrm{Spec}(A)$ *and*

$$\overline{C} = Z_1 \cup \cdots \cup Z_m$$

the unique minimal decomposition of $\overline{C}$ into irreducible and closed sets. Then for $j = 1, \ldots, m$ there exists a nonempty open subset of Z_j contained in C.

Proof: A constructible set C can be written as a finite union

$$C = L_1 \cup \cdots \cup L_{m'}$$

of nonempty, locally closed and irreducible sets L_i.

$$Z_1 \cup \cdots \cup Z_m = \overline{C} = \overline{L_1} \cup \cdots \cup \overline{L_{m'}}$$

Fix a $j \in \{1, \ldots, m\}$. As Z_j is irreducible there exists an $i \in \{1, \ldots, m'\}$ such that $Z_j \subset \overline{L_i}$. Similarly, as $\overline{L_i}$ is irreducible there exist a $j' \in \{1, \ldots, m\}$ such that $\overline{L_i} \subset Z_{j'}$. Hence

$$Z_j \subset \overline{L_i} \subset Z_{j'}.$$

This yields $j = j'$ and $Z_j = \overline{L_i}$. So L_i is a nonempty open subset of Z_j contained in C. $\square$

Lemma 6. *Let L be a locally closed and irreducible subset of* $\mathrm{Spec}(A)$*. For a constructible subset C of* $\mathrm{Spec}(A)$ *which is contained in L we have $\overline{C} = \overline{L}$ if and only if C contains the generic point of L.*

Proof: If C contains the generic point $\mathfrak{p}$ of L we have $\overline{L} = \overline{\{\mathfrak{p}\}} \subset \overline{C}$. Hence by assumption $\overline{L} = \overline{C}$.

Conversely if $\overline{C} = \overline{L}$ by Lemma 5 we know that there exists a nonempty open subset U of $\overline{L}$ contained in C. As $U \cap L$ is a nonempty open subset of L we have

$$\mathfrak{p} \in U \cap L \subset C.$$

□

Theorem 10. *Let $L \subset$ $\mathrm{Spec}(A)$ be a locally closed set and $\mathcal{G}$ an irreducible Gröbner cover of L. The following are equivalent:*

(1) *$\mathcal{G}$ is small.*

(2) *Every $Y \in \mathcal{G}$ is the only element of $\mathcal{G}$ containing the generic point of Y.*

(3) *For $Y, Y' \in \mathcal{G}$ with $Y \neq Y'$ and $Y \subset \overline{Y'}$ we have $Y \cap Y' = \emptyset$.*

Proof: The equivalence of (1) and (2) follows from lemma 6.

For two distinct, locally closed and irreducible subsets Y and Y' of $\mathrm{Spec}(A)$ the generic point of Y is contained in Y' if and only if $Y \subset \overline{Y'}$ and $Y \cap Y' \neq \emptyset$. Therefore (3) is equivalent to (2). □

Now we are prepared to prove the main theorem.

Theorem 11. *Let L be a locally closed subset of* $\mathrm{Spec}(A)$*. Then there exists exactly one irreducible, small and locally maximal Gröbner cover of L.*

Proof: First we will construct a Gröbner cover $\mathcal{G}$ of L and prove that it has the desired properties. Then we will prove uniqueness. We construct $\mathcal{G}$ recursively:

Set $C_1 = L$ and $i = 1$.
($\star$) Let

$$\overline{C_i} = Z_{i1} \cup \cdots \cup Z_{im_i}$$

be the unique minimal decomposition of $\overline{C_i}$ into irreducible and closed sets. For $j = 1, \ldots, m_i$ define

$$Y_{ij} = Z_{ij,gen} \cap \big(\text{union of all open subsets of } Z_{ij} \text{ contained in } L\big)$$

and

$$C_{i+1} = C_i \setminus (Y_{i1} \cup \cdots \cup Y_{im_i}).$$

If $C_{i+1} \neq \emptyset$ replace i by $i + 1$ and go to ($\star$).

This yields a sequence of constructible sets C_i with

$$L = C_1 \supset C_2 \supset \cdots .$$

To prove termination we will show that the sequence

$$\overline{C_1} \supset \overline{C_2} \supset \cdots$$

is strictly decreasing. For $i \geq 1$ and $j = 1, \ldots, m_i$ there exists a nonempty open subset of Z_{ij} contained in $C_i \subset L$ by lemma 5. Hence Y_{ij} is a nonempty open subset of Z_{ij} contained in L.

$$\begin{aligned}
\overline{C_{i+1}} &= \overline{C_i \setminus (Y_{i1} \cup \cdots \cup Y_{im_i})} \subset \overline{Z_{i1} \cup \cdots \cup Z_{im_i} \setminus Y_{i1} \cup \cdots \cup Y_{im_i}} \\
&\subset \overline{(Z_{i1} \setminus Y_{i1}) \cup \cdots \cup (Z_{im_i} \setminus Y_{im_i})} = (Z_{i1} \setminus Y_{i1}) \cup \cdots \cup (Z_{im_i} \setminus Y_{im_i}) \\
&\subsetneqq Z_{i1} \cup \cdots \cup Z_{im_i} = \overline{C_i}
\end{aligned}$$

This shows that there exists a (minimal) $r \in \mathbb{N}$ such that $C_{r+1} = \emptyset$. Hence

$$\begin{aligned}
\emptyset = C_{r+1} &= C_r \setminus (Y_{r1} \cup \cdots \cup Y_{rm_r}) \\
&= C_{r-1} \setminus (Y_{r-1,1} \cup \cdots \cup Y_{r-1,m_{r-1}} \cup Y_{r1} \cup \cdots \cup Y_{rm_r}) = \cdots \\
&= C_1 \setminus (Y_{11} \cup \cdots \cup Y_{1m} \cup \cdots \cup Y_{r1} \cup \cdots \cup Y_{rm_r}).
\end{aligned}$$

So we obtain

$$L = C_1 = Y_{11} \cup \cdots \cup Y_{1m} \cup \cdots \cup Y_{r1} \cup \cdots \cup Y_{rm_r}.$$

As the Y_{ij}'s are parametric by construction this shows that

$$\mathcal{G} = \big\{(Y_{ij}, G_{Y_{ij}})\ ;\ 1 \leq i \leq r,\ 1 \leq j \leq m_i\big\}$$

is a Gröbner cover of L. It is clearly irreducible. Next we will show that $\mathcal{G}$ is locally maximal. So let $Y \subset L$ be parametric with

$$Y_{ij} \subset Y \subset \overline{Y_{ij}} = Z_{ij}.$$

Then Y is an open parametric subset of Z_{ij} and so by theorem 9 we have $Y \subset Z_{ij,gen}$. From the definition of Y_{ij} we obtain $Y \subset Y_{ij}$ and thus $Y = Y_{ij}$.

Now we will show that $\mathcal{G}$ is small. Let $Y_{ij}, Y_{i'j'} \in \mathcal{G}$ with $(i,j) \neq (i',j')$.

We want to show that for $i \leq i'$ we have $Y_{ij} \not\subset \overline{Y_{i'j'}}$. Assume the contrary. Then

$$\overline{Y_{i'j'}} = Z_{i'j'} \subset \overline{C_{i'}} \subset \overline{C_i} = Z_{i1} \cup \cdots \cup Z_{im_i}.$$

Consequently there exists an $l \in \{1, \ldots, m_i\}$ such that $Z_{i'j'} \subset Z_{il}$. This yields

$$Z_{ij} = \overline{Y_{ij}} \subset \overline{Y_{i'j'}} = Z_{i'j'} \subset Z_{il}.$$

Therefore $j = l$ and $Z_{ij} = Z_{i'j'}$. For $i = i'$ this directly gives the contradiction $j = j'$. For $i < i'$ we have

$$Z_{ij} = Z_{i'j'} \subset \overline{C_{i'}} \subset \overline{C_{i+1}} \subset (Z_{i1} \setminus Y_{i1}) \cup \cdots \cup (Z_{im_i} \setminus Y_{im_i}).$$

Consequently $Z_{ij} \subset Z_{ij} \setminus Y_{ij}$ and we obtain the contradiction $Y_{ij} = \emptyset$.

To prove that $\mathcal{G}$ is small it suffices, by theorem 10, to show that for $i > i'$ and $Y_{ij} \subset \overline{Y_{i'j'}}$ we have $Y_{ij} \cap Y_{i'j'} = \emptyset$. Note that $Y_{ij} \subset \overline{Y_{i'j'}}$ implies that $Z_{ij} \setminus Y_{i'j'}$ is a closed subset of $\mathrm{Spec}(A)$. By construction we have

$$C_i = C_{i'} \setminus \big(Y_{i'1} \cup \cdots \cup Y_{i'm_{i'}} \cup \cdots \cup Y_{i-1,1} \cup \cdots \cup Y_{i-1,m_{i-1}}\big). \tag{2}$$

For subsets B, C, D of an arbitrary topological space with $D \subset C$ there is the trivial identity

$$\overline{\overline{B \setminus C} \setminus D} = \overline{B \setminus C}.$$

Together with (2) this yields

$$\begin{aligned}\overline{C_i} &= \overline{\overline{C_i} \setminus Y_{i'j'}} = \overline{Z_{i1} \cup \cdots \cup Z_{im_i} \setminus Y_{i'j'}} \subset \overline{Z_{i1} \cup \cdots \cup (Z_{ij} \setminus Y_{i'j'}) \cup \cdots \cup Z_{im_i}} \\ &= Z_{i1} \cup \cdots \cup (Z_{ij} \setminus Y_{i'j'}) \cup \cdots \cup Z_{im_i} \subset Z_{i1} \cup \cdots \cup Z_{im_i} = \overline{C_i}.\end{aligned}$$

Therefore

$$Z_{i1} \cup \cdots \cup Z_{im_i} = Z_{i1} \cup \cdots \cup (Z_{ij} \setminus Y_{i'j'}) \cup \cdots \cup Z_{im_i}$$

and $Z_{ij} \subset Z_{ij} \setminus Y_{i'j'}$. Thus $Y_{ij} \cap Y_{i'j'} = \emptyset$.

So far we have shown that $\mathcal{G}$ is an irreducible, small and locally maximal Gröbner cover of L. It remains to prove uniqueness. Assume $\mathcal{G}'$ is another irreducible, small and locally maximal Gröbner cover of L. First we will show $\mathcal{G} \subset \mathcal{G}'$. More precisely we will show, by induction on $i = 1, \ldots, r$, that $Y_{i1}, \ldots, Y_{im_i} \in \mathcal{G}'$. We denote the generic point of Y_{ij} by $\mathfrak{p}_{ij}$.

First assume $i = 1$. Let $j \in \{1, \ldots, m_1\}$. As

$$\bigcup_{Y \in \mathcal{G}} Y = L = \bigcup_{Y' \in \mathcal{G}'} Y'$$

there exists a $Y'_{1j} \in \mathcal{G}'$ such that $\mathfrak{p}_{1j} \in Y'_{1j}$. We want to show $Y_{1j} = Y'_{1j}$. As Y'_{1j} is irreducible and $\overline{Y'_{1j}} \subset \overline{L} = Z_{11} \cup \cdots \cup Z_{1m_1}$ there exist a $j' \in \{1, \ldots, m_1\}$ such that $\overline{Y'_{1j}} \subset Z_{1j'}$. Together with $\mathfrak{p}_{1j} \in Y'_{1j}$ this gives

$$Z_{1j} \subset \overline{Y'_{1j}} \subset Z_{1j'}.$$

Therefore $j = j'$ and $\overline{Y'_{1j}} = Z_{1j}$. Thus Y'_{1j} is an open subset of Z_{1j} contained in L and by theorem 9 $Y'_{1j} \subset Z_{1j,gen}$. So by definition of Y_{1j} we have $Y'_{1j} \subset Y_{1j}$. Since $\mathcal{G}'$ is locally maximal we obtain $Y_{1j} = Y'_{1j} \in \mathcal{G}'$.

Now we do the induction step. Suppose

$$Y_{11}, \ldots, Y_{1m_1}, \ldots, Y_{i-1,1}, \ldots, Y_{i-1,m_{i-1}} \in \mathcal{G}'.$$

We have to show $Y_{i1}, \ldots, Y_{im_i} \in \mathcal{G}'$. For $j \in \{1, \ldots, m_i\}$ there exists a $Y'_{ij} \in \mathcal{G}'$ such that $\mathfrak{p}_{ij} \in Y'_{ij}$. Using that $\mathcal{G}'$ is small and the induction hypothesis we obtain

$$\overline{Y'_{ij}} = \overline{Y'_{ij} \setminus \bigcup_{Y' \in \mathcal{G}' \setminus \{Y'_{ij}\}} Y'} \subset \overline{L \setminus \bigcup_{\substack{1 \le i' \le i-1 \\ 1 \le j' \le m_{i'}}} Y_{i'j'}} = \overline{C_i} = Z_{i1} \cup \cdots \cup Z_{im_i}.$$

Hence there exists a $j' \in \{1, \ldots, m_i\}$ such that $\overline{Y'_{ij}} \subset Z_{ij'}$. Together with $\mathfrak{p}_{ij} \in Y'_{ij}$ this gives

$$Z_{ij} \subset \overline{Y'_{ij}} \subset Z_{ij'}.$$

Therefore $j = j'$ and $\overline{Y'_{ij}} = Z_{ij}$. Since $\mathcal{G}'$ is locally maximal a similar argument as in the case $i = 1$ above proves $Y_{ij} = Y'_{ij} \in \mathcal{G}'$. Thus we have shown $\mathcal{G} \subset \mathcal{G}'$.

Assume this is a proper inclusion. Then there exist a $Y' \in \mathcal{G}'$ such that $Y' \notin \mathcal{G}$ and therefore

$$\overline{Y'} = \overline{Y' \smallsetminus \bigcup_{Y \in \mathcal{G}' \smallsetminus \{Y'\}} Y} \subset \overline{Y' \smallsetminus \bigcup_{Y \in \mathcal{G}} Y} = \overline{Y' \smallsetminus L} = \emptyset.$$

This is a contradiction as, by definition, the empty set is not irreducible. $\square$

Definition 13. *Let L be a locally closed subset of* $\mathrm{Spec}(A)$*. The uniquely determined irreducible, small and locally maximal Gröbner cover of L is called the* canonical irreducible Gröbner cover of L *(with respect to I and $<$).*

In [22] Weispfenning gave a rather ad hoc kind of construction for what he called canonical Gröbner systems. This construction bears some analogy with the existence proof of the above theorem, however there are some differences between the concept of canonical Gröbner systems and the concept of canonical irreducible Gröbner covers. For example, the canonical Gröbner system may contain redundant elements. The persistent reader is invited to verify this with the example $A = k[u_1, u_2]$ and $I = \langle u_1 u_2, u_1 x^2 + x \rangle$. (The point is simply that if $\mathrm{Spec}(A) = Z_1 \cup \cdots \cup Z_m$ is the decomposition of $\mathrm{Spec}(A)$ into irreducible closed sets, then it may happen that the singular part of Z_i ($= Z_i \smallsetminus Z_{i,gen}$) is contained in some $Z_{j,gen}$.)

Note that theorem 11 implies that the equivalence relation on $\mathrm{Spec}(A)$, given by comparing the leading terms of $\langle \sigma_{\mathfrak{p}}(I) \rangle$, has only finitely many equivalence classes and that every equivalence class is a constructible set. Indeed example 2 and 10 show that these equivalence classes are only constructible and not locally closed. The following example illustrates that the canonical irreducible Gröbner cover may be not of minimal cardinality among the irreducible Gröbner covers.

Example 10. Let k be a field and $A = k[u_1, u_2]$ the polynomial ring in the two parameters u_1, u_2. We consider the ideal

$$I = \langle u_1 x, (u_2^2 - 1)x^2 + x \rangle \subset A[x].$$

(Here x denotes just one variable.) Obviously $J = J(I) = \langle u_1 \rangle$ and the affine plane without the u_2-axis has generic Gröbner basis x, i.e. $Y_1 = \mathbb{A}^2_{gen} = \mathrm{Spec}(A) \smallsetminus \mathrm{V}(u_1)$ and $x \in \mathcal{I}_{Y_1}(Y_1) = I_{u_1}$ ($=$ localization of I at $\{1, u_1, u_1^2, \ldots\}$) is the reduced Gröbner basis of I over Y_1. By factoring mod $J = \langle u_1 \rangle$ and identifying A/J with $k[u_2]$ we obtain

$$\overline{I} = \langle (u_2^2 - 1)x^2 + x \rangle \subset k[u_2][x].$$

On the u_2-axis the generic Gröbner basis is $x^2 + \frac{1}{u_2^2 - 1} x$, i.e.

$$J(\overline{I}) = \langle u_2^2 - 1 \rangle = \langle u_2 + 1 \rangle \cap \langle u_2 - 1 \rangle,$$

$Y_2 = \mathrm{V}(u_1)_{gen} = \mathrm{V}(u_1) \smallsetminus \mathrm{V}(u_2^2 - 1)$ and $x^2 + \frac{1}{u_2^2 - 1} x \in \mathcal{I}_{Y_2}(Y_2) = \overline{I}_{u_2^2 - 1}$ is the reduced Gröbner bases of I over Y_2. Finally over the two closed points $Y_3 = \langle u_1, u_2 - 1 \rangle$ and $Y_4 = \langle u_1, u_2 + 1 \rangle$ we have the reduced Gröbner basis x again. To summarize

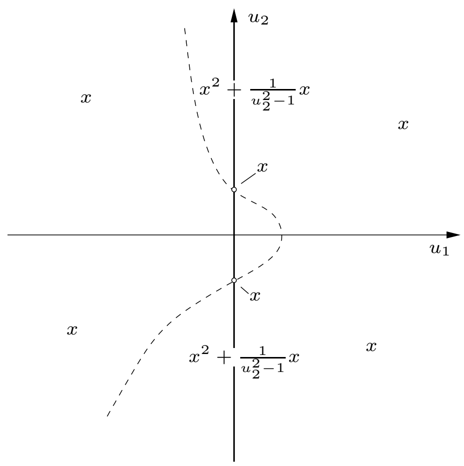
$$\mathcal{G} = \big\{ (Y_1, \{x\}), (Y_2, \{x^2 + \tfrac{1}{u_2^2 - 1} x\}), (Y_3, \{x\}), (Y_4, \{x\}) \big\}$$

is the canonical irreducible Gröbner cover of $\mathbb{A}^2 = \mathrm{Spec}(A)$.

Let $f \in k[u_1, u_2]$ be an irreducible polynomial such that $f(0, u_2) = u_2^2 - 1$ (e.g. $f = u_1 + u_2^2 - 1$). Then there exist $h \in A = k[u_1, u_2]$ such that $f = hu_1 + u_2^2 - 1$, thus $fx^2 + x = (hx)(u_1x) + (u_2^2 - 1)x^2 + x \in I$. Therefore the extension of I in $(A/\langle f\rangle)[x]$ is just $\langle x\rangle$ and $\mathrm{V}(f)$ is parametric with reduced Gröbner basis x. Consequently

$$\mathcal{G}' = \{(Y_1, \{x\}), (Y_2, \{x^2 + \tfrac{1}{u_2^2-1}x\}), (\mathrm{V}(f), x)\}$$

is an irreducible Gröbner cover of $\mathbb{A}^2$ with smaller cardinality than the canonical irreducible Gröbner cover. However, choosing an irreducible Gröbner cover of $\mathrm{Spec}(A)$ with minimal cardinality in a canonical way is as impossible as choosing a curve which meets the u_2-axes only in $(0, -1)$ and $(0, 1)$ in a canonical way.



The above example can also be used to show that a parametric subset of $\mathrm{Spec}(A)$ need not be contained in a maximal parametric subset.

3.1 The projective case

In the projective setting, i.e. if I is a homogeneous ideal the situation is considerably nicer than in the affine setting. It actually is as nice as it can be hoped for: The equivalence classes of the equivalence relation $\sim$ defined on $\mathrm{Spec}(A)$ by $\mathfrak{p} \sim \mathfrak{p}'$ if $\mathrm{lt}(\langle\sigma_{\mathfrak{p}}(I)\rangle) = \mathrm{lt}(\langle\sigma_{\mathfrak{p}'}(I)\rangle)$ are parametric. (In particular they are locally closed.) The key to the proof is the following lemma which is not true for arbitrary ideals (cf. example 3 and 10). The equivalence of (1) and (2) has already been proved for $A = \mathbb{Z}$ in [2] (theorem 5.13).

Lemma 7. *Let $I \subset A[x]$ be a homogeneous ideal and $\mathfrak{p} \in \mathrm{Spec}(A)$. Then the following assertions are equivalent:*

(1) *$\mathfrak{p}$ is lucky for I.*

(2) $\mathrm{lt}(\langle\sigma_{\mathfrak{p}}(I)\rangle) = \mathrm{lt}(I)$.

(3) $\mathrm{lt}(\langle\sigma_{\mathfrak{p}}(I)\rangle) \supset \mathrm{lt}(I)$.

Proof: We have already seen that (1) implies (2) in theorem 9. So we only have to show that (3) implies (1):

Assume that $\mathfrak{p} \in \mathrm{Spec}(A)$ is unlucky for I. Then there exists $t \in \mathrm{lt}(I)$ such that $\mathrm{lc}(I,t) \subset \mathfrak{p}$. We may assume that t is maximal in its degree, i.e. for every $t' \in \mathrm{lt}(I)$ with $\deg(t') = \deg(t)$ and $\mathrm{lc}(I,t') \subset \mathfrak{p}$ we have $t' \leq t$. Since $t \in \mathrm{lt}(I) \subset \mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ there exists $P \in I$ such that $\mathrm{lt}(\sigma_{\mathfrak{p}}(P)) = t$. Because I is homogeneous we may assume that P is homogeneous and thus $\deg(P) = \deg(t)$. We can also assume that $\mathrm{lt}(P)$ is minimal, i.e. for $P' \in I$ with $\mathrm{lt}(\sigma_{\mathfrak{p}}(P')) = t$ we have $\mathrm{lt}(P') \geq \mathrm{lt}(P)$.

Because $\mathrm{lc}(I,t) \subset \mathfrak{p}$ we have $\mathrm{lt}(P) > t$. By the maximality of t we conclude $\mathrm{lc}(I, \mathrm{lt}(P)) \not\subseteq \mathfrak{p}$. Thus there exists $Q \in I$ with $\mathrm{lt}(Q) = \mathrm{lt}(P)$ and $\mathrm{lc}(Q) \notin \mathfrak{p}$. Set

$$P' = \mathrm{lc}(Q)P - \mathrm{lc}(P)Q.$$

Then for $t' > t$ we have $\mathrm{coef}(P', t') \in \mathfrak{p}$ because $\mathrm{coef}(P, t'), \mathrm{lc}(P) \in \mathfrak{p}$. On the other hand $\mathrm{coef}(P', t)$ does not lie in $\mathfrak{p}$ because $\mathrm{lc}(Q), \mathrm{coef}(P, t) \notin \mathfrak{p}$. Therefore $\mathrm{lt}(\sigma_{\mathfrak{p}}(P')) = t$ but as $\mathrm{lt}(P') < \mathrm{lt}(P)$ this contradicts the minimality of P'. $\square$

Note that if $I \subset A[x]$ is an arbitrary ideal and $\mathfrak{p} \in \mathrm{Spec}(A)$ is unlucky for I then we can say virtually nothing about the relation between $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ and $\mathrm{lt}(I)$. We may have $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle) \subsetneqq \mathrm{lt}(I)$. (This for example happens if I is a monomial ideal.) Or we may have $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle) \supsetneqq \mathrm{lt}(I)$. (This for example happens if I is generated by a single polynomial $P = \sum_{i=1}^{m} a_i t_i$ such that t_i divides t_{i+1} and the a_i's generate the unit ideal in A.) It may also happen that $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ and $\mathrm{lt}(I)$ are incomparable, i.e. there does not hold any inclusion relation between them. Finally it may actually happen that $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ equals $\mathrm{lt}(I)$ (see example 3).

By the above lemma, we at least know that $\mathrm{lt}(I)$ is not contained in $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ if I is homogeneous and $\mathfrak{p}$ unlucky for I.

Theorem 12. *Let $I \subset A[x]$ be a homogeneous ideal and $L \subset \mathrm{Spec}(A)$ locally closed. Then the equivalence classes of the equivalence relation $\sim$ defined on L by $\mathfrak{p} \sim \mathfrak{p}'$ if $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle) = \mathrm{lt}(\langle \sigma_{\mathfrak{p}'}(I) \rangle)$ are parametric with respect to I.*

Proof: By theorem 3 every locally closed subset of a parametric subset is parametric. Thus we may assume $L = \mathrm{Spec}(A)$. Let $Y \subset \mathrm{Spec}(A)$ be an equivalence class and $T' \subset \mathcal{T}$ such that $\mathrm{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle) = T'$ for all $\mathfrak{p} \in Y$. From theorem 11 we already know that Y is a constructible subset of $\mathrm{Spec}(A)$. Let Z be the closure of Y and $\mathfrak{a} \subset A$ the radical ideal such that $\overline{Y} = Z = \mathrm{V}(\mathfrak{a})$. As usual $\overline{I}$ denotes the extension of I in $(A/\mathfrak{a})[x]$. To apply lemma 7 we have to show $\mathrm{lt}(\overline{I}) = T'$. Let

$$Z = Z_1 \cup \cdots \cup Z_m$$

be the unique minimal decomposition of Z into irreducible and closed subsets. For $i = 1, \ldots, m$ let $\mathfrak{a}_i \subset A$ denote the radical ideal such that $Z_i = \mathrm{V}(\mathfrak{a}_i)$ and $\overline{I}_i$ the extension of I in $(A/\mathfrak{a})[x]$. By lemma 5 the intersection $Z_{i,gen} \cap Y$ is nonempty. Therefore by theorem 9 we have $\mathrm{lt}(\overline{I}_i) = \mathrm{lt}(Z_{i,gen}) = T'$.

Now let $P \in \overline{I}$. If for every $i \in \{1, \ldots, m\}$ the leading term of the image of P in $\overline{I}_i$ is strictly smaller than the leading term of P, then the leading coefficient of P must lie in the intersection of all the $\mathfrak{a}_i$'s which is zero mod $\mathfrak{a}$. Thus there exists $i \in \{1, \ldots, m\}$ such that $\mathrm{lt}(P) \in \mathrm{lt}(\overline{I}_i) = T'$. Consequently $\mathrm{lt}(\overline{I}) \subset T'$.

For the converse direction let $t \in T' = \operatorname{lt}(\overline{I}_1)$. There exists $P \in \overline{I}$ such that the leading term of the image of P in $\overline{I}_1$ is t. This means $\operatorname{coef}(P, t') \in \mathfrak{a}_1$ for $t' > t$ and $\operatorname{coef}(P, t) \notin \mathfrak{a}_1$. The $\mathfrak{a}_i$'s constitute the minimal primary decomposition of $\mathfrak{a}$ and so we can find $c \in \mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_m \setminus \mathfrak{a}_1$. For $t' > t$ the coefficient of cP at t' lies in the intersection of all the $\mathfrak{a}_i$'s and thus equals zero. On the other hand $\operatorname{coef}(cP, t)$ does not lie in $\mathfrak{a}_1$ and therefore $\operatorname{lt}(cP) = t$. Consequently $t \in \operatorname{lt}(\overline{I})$.

By definition Y is the set of all primes $\mathfrak{p} \in Z$ such that $\operatorname{lt}(\langle \sigma_{\mathfrak{p}}(I) \rangle)$ equals $T' = \operatorname{lt}(\overline{I})$. Thus, by lemma 7, Y is the set of all lucky primes of $\overline{I}$, i.e. $Y = Z_{gen}$ which is parametric by proposition 8. $\square$

It is now obvious how to define the canonical Gröbner cover in the projective case:

Definition 14. *Let I be a homogeneous ideal of $A[x]$ and L a locally closed subset of* $\operatorname{Spec}(A)$. *The Gröbner cover corresponding to the stratification of L with respect to the leading terms of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ is called the* canonical Gröbner cover of L with respect to I (and $<$).

Conclusion and open questions

We have introduced two concepts for studying the geometry of fibres: parametric sets and Gröbner covers. It seems possible to generalize these notions to more general (i.e. not necessarily affine) base schemes.

Clearly one of the main reasons for the success of Gröbner bases in the last decades has been the fact that in many cases they could actually be computed. The focus of this article was not on algorithms, but of course an efficient implementation of an algorithm to compute Gröbner covers is desirable. The existence proof for the canonical irreducible Gröbner cover is in principle constructive, but an algorithm for the computation of the canonical irreducible Gröbner cover would necessarily involve successive primary decompositions and thus would be of modest practical value. The obvious solution is to skip irreducibility. For the projective case we have the canonical Gröbner cover at hand and it suggests itself to exploit this for the affine case by a process of homogenizing and dehomogenizing.

The problem of determining the Gröbner basis structure of the fibres has already been considered from an algorithmic point of view (see [16], [17], [22], [21]). Most notably Antonio Montes released an implementation in Maple (see `http://www-ma2.upc.edu/~montes`) for the important case where A is the polynomial ring over $\mathbb{Q}$. In fact, the output of his algorithm BUILDTREE can be interpreted as a Gröbner cover, but a drawback is that you cannot say a priori which Gröbner cover the algorithm will compute, furthermore the result depends on a term order on the parameters.

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